CALCULUS AB
REVIEW WORKSHEET ON DIFFERENTIAL EQUATIONS

Work these on notebook paper. Do not use your calculator.

Solve for y as a function of x.

1. $\dfrac{dy}{dx} = x^2(y-1)$ $\quad y(0) = 3$

2. $\dfrac{dy}{dx} = 4y^2 \sec^2(2x)$ $\quad y\left(\dfrac{\pi}{8}\right) = 1$

3. $\dfrac{dy}{dx} = 8xy^2$ $\quad y(-1) = -\dfrac{1}{3}$

4. $\dfrac{dy}{dx} = -\dfrac{2x}{y}$ $\quad y(1) = -1$

5. $xy\dfrac{dy}{dx} = \ln x$ $\quad y(1) = -4$

6. $\dfrac{dy}{dx} = \dfrac{y-3}{x^2}$ $\quad y(4) = 0$

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7. Consider the differential equation $\dfrac{dy}{dx} = \dfrac{x}{y}$ .Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(-5) = -1$,

$\int y\,dy = \int x\,dx$

$\dfrac{y^2}{2} = \dfrac{x^2}{2} + C$

$\dfrac{1}{2} = \dfrac{25}{2} + C$

$C = -12$

$\dfrac{y^2}{2} = \dfrac{x^2}{2} - 12$

$y^2 = x^2 - 24$

$y = \pm\sqrt{x^2-24}$

* a) $\boxed{y = -\sqrt{x^2-24}}$

* b) $\boxed{\begin{array}{c} y+1 = 5(x+5) \\ f(-4.9) \approx -.5 \end{array}}$

Write a differential equation to represent the following:

8. The rate of change of a population y, with respect to time t, is proportional to t.

$\dfrac{dy}{dt} = kt$

9. The rate of change of a population P, with respect to time t, is proportional to the cube of the population.

$\dfrac{dP}{dt} = k \cdot P^3$

10. Water leaks out of a barrel at a rate proportional to the square root of the depth of the water at that time.

$\dfrac{dw}{dt} = k \cdot \sqrt{\ell}$ $\qquad \ell$ = depth

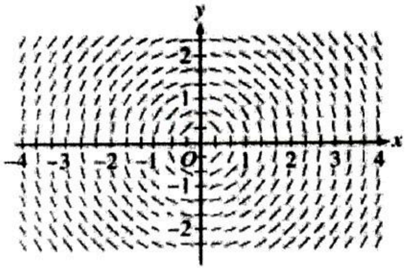
11. Oil leaks out of a tank at a rate inversely proportional to the amount of oil in the tank.

$\dfrac{dO}{dt} = \dfrac{k}{(C-O)}$

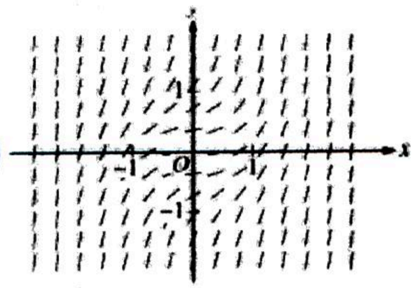
O = oil leaking out
C = amount of oil in tank @ t = 0

12. Which of the following is a slope field for the differential equation $\frac{dy}{dx} = x^2 + y^2$?

~~A~~



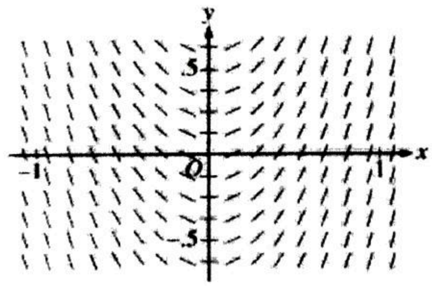
B



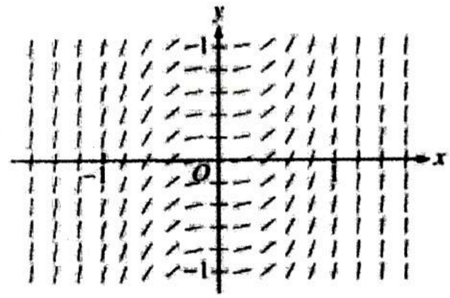
$\frac{dy}{dx} \geq 0$

for all x

~~C~~



~~D~~



$\left.\frac{dy}{dx}\right|_{(1,0)} \neq 0$

13.

Oil is being pumped continuously from a certain oil well at a rate proportional to the amount of oil left in the well; that is, $dy/dt=ky$, where y is the amount of oil left in the well at any time t. Initially there were 1,000,000 gallons of oil in the well, and 6 years later there were 500,000 gallons remaining. It will no longer be profitable to pump oil when there are fewer than 50,000 gallons remaining.

Write an equation for y, the amount of oil remaining in the well at any time t.

$\frac{dy}{dt} = ky \rightarrow \quad y = Ce^{kt}$

$y = 1{,}000{,}000\, e^{kt}$

$\frac{1}{2} = e^{k \cdot 6} \rightarrow e^k = 2^{\frac{1}{6}}$

t	y
0	1,000,000
6	500,000

$\boxed{y = 1{,}000{,}000 \left(\frac{1}{2}\right)^{\frac{1}{6}t}}$

At what rate is the amount of oil in the well decreasing when there are 600,000 gallons of oil remaining?

$K = \frac{\ln\left(\frac{1}{2}\right)}{6}$

$\left.\frac{dy}{dt}\right|_{y=600{,}000} = \frac{\ln\left(\frac{1}{2}\right)}{6} \cdot 600{,}000$

$= 100{,}000 \ln\left(\frac{1}{2}\right)$ gallons/yr.

Differential Review

1. $\int \frac{dy}{y-1} = \int x^2\,dx \qquad (0,3)$

$\ln|y-1| = \frac{x^3}{3} + C$

$\ln 2 = \frac{0^3}{3} + C$

$C = \ln 2$

$e^{\ln|y-1|} = e^{\frac{x^3}{3} + \ln 2}$

$|y-1| = e^{\frac{x^3}{3} + \ln 2}$

$y-1 = \pm 2e^{x^3/3}$

$y = 1 \pm 2e^{x^3/3}$

check pt.

$\boxed{y = 1 + 2e^{x^3/3}}$

2. $\int \frac{dy}{y^2} = \int 4\sec^2(2x)\,dx \qquad u = 2x \qquad \frac{du}{dx} = 2 \qquad dx = \frac{du}{2} \qquad \left(\frac{\pi}{8}, 1\right)$

$\int y^{-2}\,dy = 4\int \sec^2(2x)\,dx$

$\frac{y^{-1}}{-1} = 2\tan(2x) + C$

$-\frac{1}{y} = 2\tan\left(\frac{\pi}{4}\right) + C$

$-1 = 2 + C$

$C = -3$

$-\frac{1}{y} = 2\tan(2x) - 3$

$\boxed{y = \frac{-1}{2\tan(2x) - 3}}$

3. $\int \frac{dy}{y^2} = \int 8x\,dx \qquad \left(-1, -\frac{1}{3}\right)$

$\int y^{-2}\,dy = 4x^2 + C$

$-\frac{1}{y} = 4x^2 + C$

$3 = 4 + C$

$C = -1$

$-\frac{1}{y} = 4x^2 - 1$

$\boxed{y = \frac{1}{1 - 4x^2}}$

4. $\int y\,dy = \int -2x\,dx \qquad (1,-1)$

$\frac{y^2}{2} = -\cancel{x^2} + C$

$\frac{1}{2} = -1 + C$

$C = 3/2$

$\frac{y^2}{2} = -x^2 + \frac{3}{2}$

$y^2 = -2x^2 + 3$

$y = \pm\sqrt{-2x^2+3}$

$\boxed{y = -\sqrt{-2x^2+3}}$

5. $\int y\,dy = \int \frac{\ln x}{x}\,dx \qquad (1,-4)$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$dx = x\,du$

$\frac{y^2}{2} = \frac{(\ln x)^2}{2} + C$

$\frac{16}{2} = \frac{0^2}{2} + C$

$C = 8$

$\frac{y^2}{2} = \frac{(\ln x)^2}{2} + 8$

$y^2 = (\ln x)^2 + 16$

$y = \pm\sqrt{(\ln x)^2+16}$

$\boxed{y = -\sqrt{(\ln x)^2+16}}$

6. $\int \frac{dy}{y-3} = \int \frac{dx}{x^2} = \int x^{-2}\,dx \qquad (4,0)$

$\ln|y-3| = -\frac{1}{x} + C$

$\ln 3 = -\frac{1}{4} + C$

$C = \frac{1}{4} + \ln 3$

$e^{\ln|y-3|} = e^{-\frac{1}{x} + \frac{1}{4} + \ln 3}$

$|y-3| = 3e^{-\frac{1}{x}+\frac{1}{4}}$

$y-3 = \pm 3e^{-\frac{1}{x}+\frac{1}{4}}$

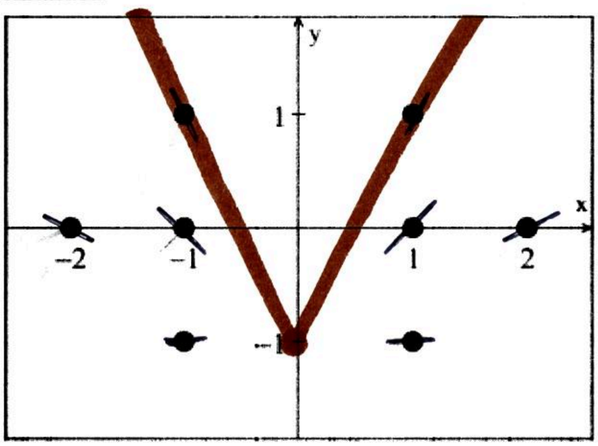
$y = 3 \pm 3e^{-\frac{1}{x}+\frac{1}{4}}$

$\boxed{y = 3 - 3e^{-\frac{1}{x}+\frac{1}{4}}}$

Free Response Non Calculator

Consider the differential equation $\frac{dy}{dx} = \frac{1+y}{x}$, where $x \neq 0$.	
A	**On the axes provided, sketch a slope field for the given differential equation at the eight points indicated.**
B	**Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(-1) = 1$ and state its domain.**

Part A:



x	y	$\frac{dy}{dx}$
1	0	1
2	0	1/2
1	1	2
1	−1	0
−1	1	−2
−1	0	−1
−1	−1	0
−2	0	−1/2

Part B:

$$\int \frac{dy}{1+y} = \int \frac{1}{x}\, dx$$

$(-1, 1)$

$$\ln|1+y| = \ln|x| + C$$

$$\ln 2 = \cancel{\ln 1}^{\,0} + C$$

$$C = \ln 2$$

$$e^{\ln|1+y|} = e^{(\ln|x| + \ln 2)}$$

$$|1+y| = e^{\ln|x|} \cdot e^{\ln 2}$$

$$|1+y| = |x| \cdot 2$$

$$1 + y = \pm 2|x|$$

$$y = -1 \pm 2|x|$$

$$y = -1 + 2|x|$$

$$\{x \mid x \in \mathbb{R}\}$$

Check:

$1 = -1 + 2|-1|$ ✓

$1 = -1 - 2|-1|$ ✗

4. At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

a) $y - 91 = \underline{-16}(t - 0)$

$H(3) \approx -16(3) + 91$

$\approx 43°C$

$\left.\frac{dH}{dt}\right|_{(0,91)} = -\frac{1}{4}(91-27) = -\frac{1}{4}(64) = -16$

b) $\frac{d^2H}{dt^2} = -\frac{1}{4}\frac{dH}{dt} = \frac{-1}{4}\left(-\frac{1}{4}(H-27)\right) = \frac{1}{16}(H-27)$

$\left.\frac{d^2H}{dt^2}\right|_{(0,91)} = \frac{1}{16}(91-27) > 0 \therefore$ H is concave up on $0 \le t \le 3$, and part (a) is an underestimate.

c) $\int \frac{dG}{(G-27)^{2/3}} = \int -dt$

$\int (G-27)^{-2/3}\, dG = \int -dt$

$3(G-27)^{1/3} = -t + C$

$3(91-27)^{1/3} = 0 + C$

$C = 3\sqrt[3]{64} = 12$

$3(G-27)^{1/3} = -t + 12$

$\sqrt[3]{G-27} = -\frac{t}{3} + 4$

* $G = \left(-\frac{t}{3} + 4\right)^3 + 27$

$G(3) = (-1+4)^3 + 27$

$= \underline{54°C}$