$\int$ Rational Functions Using Long $\div$ and Inverse Trig

Long $\div$ If your integrand is a rational function and the degree of the numerator is not less than the degree of the denominator, you might need to divide.

Example: $\int \frac{x^2+1}{x-1}\,dx$

$$\begin{array}{r} x+1+\frac{2}{x-1} \\ x-1\,\overline{)\,x^2+0x+1} \\ -(x^2-x) \\ \hline x+1 \\ -(x-1) \\ \hline 2 \end{array}$$

$= \int x\,dx + \int 1\,dx + 2\int \frac{1}{x-1}\,dx \longrightarrow$ u-sub

$u = x-1$

$\frac{du}{dx} = 1$

$du = dx$

$= \boxed{\frac{x^2}{2} + x + 2\ln|x-1| + C}$

Example: $\int \frac{x^2+x+1}{x^2+1}\,dx$

$$\begin{array}{r} 1+\frac{x}{x^2+1} \\ x^2+0x+1\,\overline{)\,x^2+x+1} \\ -(x^2+0x+1) \\ \hline x \end{array}$$

$= \int 1\,dx + \int \frac{x}{x^2+1}\,dx \longrightarrow$ u-sub

$u = x^2+1$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$\int \frac{\cancel{x}}{u}\cdot\frac{du}{2\cancel{x}}$

$= \boxed{x + \frac{1}{2}\ln(x^2+1) + C}$

Inverse Trig → Some rational functions can use inverse trig!

Reminder:

$y = \sin^{-1} u$ $\quad y' = \dfrac{du}{\sqrt{1-u^2}}$

$y = \tan^{-1} u$ $\quad y' = \dfrac{du}{1+u^2}$

$y = \sec^{-1} u$ $\quad y' = \dfrac{du}{|u|\sqrt{u^2-1}}$

$$\int \frac{1}{1+4x^2}\,dx$$

$u^2 = 4x^2$

$u = 2x$

$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

$$= \int \frac{1}{1+u^2} \cdot \frac{du}{2}$$

$$= \frac{1}{2} \int \frac{du}{1+u^2}$$

$$= \frac{1}{2} \tan^{-1} u + C$$

$$\boxed{= \frac{1}{2} \tan^{-1}(2x) + C}$$

$$\int \frac{2}{\sqrt{1-9x^2}}\,dx$$

$u^2 = 9x^2$

$u = 3x$

$\frac{du}{dx} = 3$

$dx = \frac{du}{3}$

$$= 2 \int \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{3}$$

$$= \frac{2}{3} \int \frac{du}{\sqrt{1-u^2}}$$

$$= \frac{2}{3} \sin^{-1}(3x) + C$$

$$= \boxed{\frac{2}{3} \sin^{-1}(3x) + C}$$

$$\int \frac{1/16}{\underset{/16}{x^2}+16\,/16}\,dx = \frac{1}{16}\int \frac{1}{\frac{x^2}{16}+1}\,dx$$

This needs to be a 1

$u^2 = \frac{x^2}{16}$

$u = \frac{x}{4}$

$\frac{du}{dx} = \frac{1}{4}$

$dx = du \cdot 4$

$$= \frac{1}{16}\int \frac{1}{u^2+1} \cdot du \cdot 4$$

$$= \frac{1}{4}\int \frac{1}{1+u^2}\,du$$

$$= \frac{1}{4}\tan^{-1} u + C = \boxed{\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + C}$$

$$\int \frac{1}{x\sqrt{4x^2-1}}\,dx$$

Looks like $\sec^{-1}$

$u^2 = 4x^2$

$u = 2x$

$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

$$= \int \frac{1}{x\sqrt{u^2-1}} \cdot \frac{du}{2}$$

since $u = 2x$, $x = \frac{u}{2}$

$$= \frac{1}{2}\int \frac{1}{\frac{u}{2}\sqrt{u^2-1}}\,du$$

$$= \frac{1}{\cancel{2}}\int \frac{\cancel{2}}{u\sqrt{u^2-1}}\,du$$

$$= \sec^{-1}(u) + C$$

$$= \boxed{\sec^{-1}(2x) + C}$$

$$\int \frac{1}{x^2-4x+7}\,dx$$

Time to complete the square!

$$(x^2-4x+\underline{4})+7-\underline{4}$$

$$\left(-\tfrac{4}{2}\right)^2$$

$$(x-2)^2+3$$

$$= \int \frac{1/3}{\frac{(x-2)^2}{3}+\frac{3}{3}}\,dx$$

$$= \frac{1}{3}\int \frac{1}{\frac{(x-2)^2}{3}+1}\,dx$$

Now it looks like $\tan^{-1}u$!

$$u^2 = \frac{(x-2)^2}{3}$$

$$u = \frac{x-2}{\sqrt{3}} = \frac{1}{\sqrt{3}}x - \frac{2}{\sqrt{3}}$$

$$= \frac{1}{3}\int \frac{\sqrt{3}\,du}{u^2+1}$$

$$\frac{du}{dx} = \frac{1}{\sqrt{3}}$$

$$dx = \sqrt{3}\cdot du$$

$$= \frac{\sqrt{3}}{3}\tan^{-1}(u)+C$$

$$\boxed{= \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{x-2}{\sqrt{3}}\right)+C}$$