Limits – Indeterminate Forms and L'Hospital's Rule

Suppose that f and g are differentiable functions on an open interval containing $x = a$, except possibly at $x = a$, and that $\lim\limits_{x\to a} f(x) = 0$ or $\pm\infty$ and $\lim\limits_{x\to a} g(x) = 0$ or $\pm\infty$.

If $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ has a finite limit, or if this limit is $+\infty$ or $-\infty$, then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$.

*Moreover, this statement is also true in the case of a limit as $x \to a^-, x \to a^+, x \to -\infty,$ or as $x \to +\infty$.

Step 1. Check that the limit of $\dfrac{f(x)}{g(x)}$ is an indeterminate form of type $\dfrac{0}{0}$ **or** $\dfrac{\infty}{\infty}$.

*If it is not, then **L'Hospital's Rule** cannot be used.*

Step 2. Differentiate f and g separately.

*[Note: **Do not differentiate** $\dfrac{f(x)}{g(x)}$ **using the quotient rule!**]*

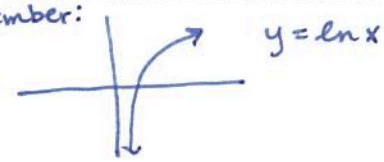
Step 3. Find the limit of $\dfrac{f'(x)}{g'(x)}$. If this limit is finite, $+\infty$, or $-\infty$, then it is equal to the limit of $\dfrac{f(x)}{g(x)}$.

Step 4. If the limit is still an indeterminate form of type $\dfrac{0}{0}$ **or** $\dfrac{\infty}{\infty}$, then simplify $\dfrac{f'(x)}{g'(x)}$ algebraically and apply **L'Hospital's Rule** again.

I. Indeterminate Form of the Type $\frac{0}{0}$.

Example 1:	Example 2:	Example 3:
$\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2} =$	$\lim\limits_{x\to 0}\dfrac{\tan 3x}{\sin 2x} =$	$\lim\limits_{x\to 0}\dfrac{e^x - x - 1}{x^2} =$
$\lim\limits_{x\to 2}(x^2-4)=0$	$\lim\limits_{x\to 0}\tan 3x = 0$	$\lim\limits_{x\to 0}(e^x-x-1)=0$
$\lim\limits_{x\to 2}(x-2)=0$	$\lim\limits_{x\to 0}\sin 2x = 0$	$\lim\limits_{x\to 0} x^2 = 0$
$\therefore$ L'HOP Applies	$\therefore$ L'HOP APPLIES	$\therefore$ L'HOP APPLIES
$\lim\limits_{x\to 2}\dfrac{2x}{1} = 4$	$\lim\limits_{x\to 0}\dfrac{3\sec^2 3x}{2\cos 2x} = \dfrac{3}{2}$	$\lim\limits_{x\to 0}\dfrac{e^x-1}{2x}$
		$\lim\limits_{x\to 0}(e^x-1)=0$
		$\lim\limits_{x\to 0} 2x = 0$
		$\therefore$ L'HOP APPLIES
		$\lim\limits_{x\to 0}\dfrac{e^x}{2} = \dfrac{1}{2}$

Remember:



II. Indeterminate Form of the Type $\frac{\infty}{\infty}$

Example 1:

$$\lim_{x\to+\infty}\frac{3x^2+5x-7}{2x^2-3x+1}$$

$\lim_{x\to\infty}(3x^2+5x-7)=\infty$

$\lim_{x\to\infty}(2x^2-3x+1)=\infty$

$\therefore$ L'HOP APPLIES

$\lim_{x\to\infty}\frac{6x+5}{4x-3}$

$\lim_{x\to\infty}(6x+5)=\infty$

$\lim_{x\to\infty}(4x-3)=\infty$

$\therefore$ L'HOP APPLIES

$\lim_{x\to\infty}\frac{6}{4}=\frac{3}{2}$

* which is what we get from BOBO BOTNO CO/CO !! :)

Example 2:

$$\lim_{x\to\infty}\frac{x^2+1}{e^x}$$

$\lim_{x\to\infty}(x^2+1)=\infty$

$\lim_{x\to\infty}e^x=\infty$

$\therefore$ L'HOP APPLIES

$\lim_{x\to\infty}\frac{2x}{e^x}$

$\lim_{x\to\infty}2x=\infty$

$\lim_{x\to\infty}e^x=\infty$

$\therefore$ L'HOP APPLIES

$\lim_{x\to\infty}\frac{2}{e^x}=0$

* $\frac{\text{constant}}{\infty}\to 0$

Example 3:

$$\lim_{x\to+\infty}\frac{\ln(x^2+1)}{\ln(x^3+1)}$$

$\lim_{x\to\infty}\ln(x^2+1)=\infty$

$\lim_{x\to\infty}\ln(x^3+1)=\infty$

$\therefore$ L'HOP APPLIES

$\lim_{x\to\infty}\left(\frac{\frac{2x}{x^2+1}}{\frac{3x^2}{x^3+1}}\right)=\lim_{x\to\infty}\left(\frac{2x^4+2x}{3x^4+3x^2}\right)$

$\frac{CO}{CO}! = \frac{2}{3}$

AP Calculus AB

Name Key

WKS: 8.7 L'Hôpital's Rule & Indeterminate Forms

Some of these problems require L'Hôpital's Rule, others do not. Make sure that when you use L'Hôpital's Rule, you justify its use.

1) $\lim\limits_{x\to 4}\sqrt{2x-5} = \sqrt{2(4)-5} = \sqrt{3}$

2) $\lim\limits_{x\to\infty}\dfrac{\ln x}{x} = \lim\limits_{x\to\infty}\dfrac{\frac{1}{x}}{1} = 0$

$\lim\limits_{x\to\infty}\ln x = \infty$

$\lim\limits_{x\to\infty} x = \infty$

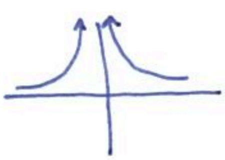
∴ L'HOP APPLIES

3) $\lim\limits_{x\to 2}\dfrac{x-2}{x^2-4} = \lim\limits_{x\to 2}\dfrac{1}{x+2} = \dfrac{1}{4}$

4) $\lim\limits_{x\to -3}\dfrac{x^2-x-12}{2x^2+5x-3} = \lim\limits_{x\to -3}\dfrac{(x-4)\cancel{(x+3)}}{(2x-1)\cancel{(x+3)}}$

$= \dfrac{-7}{-7} = 1$

5) $\lim\limits_{x\to 0}\dfrac{4}{x^2} = \infty$



6) $\lim\limits_{x\to 0}\dfrac{e^{2x}-1}{x} = \lim\limits_{x\to 0}\dfrac{2e^{2x}}{1} = 2$

$\lim\limits_{x\to 0}(e^{2x}-1) = 0$

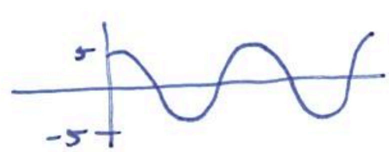
$\lim\limits_{x\to 0} x = 0$

∴ L'HOP APPLIES

7) $\lim\limits_{x\to\infty}\dfrac{3x-1}{2x^2+1} = 0$

BOBO

8) $\lim\limits_{x\to\infty}(5\cos x) =$ DNE



9) $\lim_{x \to \infty} \frac{3x^3 - 1}{2x^2 + 1} = \infty$

BOTNO

10) $\lim_{x \to \pi} \frac{\sin x}{1 - \cos x} = \lim_{x \to \pi} \frac{\cos x}{\sin x} = \lim_{x \to \pi} \cot x =$ DNE

$\lim_{x \to \pi} \sin x = 0$

$\lim_{x \to \pi} (1 - \cos x) = 0$

$\therefore$ L'HOP APPLIES

$\lim_{x \to \pi} \cos x = 1$ $\therefore$ L'HOP DOES NOT APPLY TWICE

11) $\lim_{\substack{\cancel{x \to -3} \\ x \to \infty}} \frac{x^2 - x - 12}{2x^2 + 5x - 3} = \frac{1}{2}$

Repeat of #4

Change $x \to -3$

to $x \to \infty$

CO/CO

12) $\lim_{x \to 0} \frac{\sin 6x}{4x}$

$\lim_{x \to 0} \sin 6x = 0$

$\lim_{x \to \infty} 4x = 0$

$\therefore$ L'HOP APPLIES

$\lim_{x \to 0} \frac{6 \cos 6x}{4} = \frac{3}{2}$

13) $\lim_{x \to 2} \frac{x^3 - 8}{x - 2} = \lim_{x \to 2} \frac{3x^2}{1} = 12$

$\lim_{x \to 2} (x^3 - 8) = 0$

$\lim_{x \to 2} (x - 2) = 0$

$\therefore$ L'HOP APPLIES

14) $\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \frac{\frac{1}{x}}{1} = 1$

$\lim_{x \to 1} \ln x = 0$

$\lim_{x \to 1} (x - 1) = 0$

$\therefore$ L'HOP APPLIES

15) $\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{e^x}{2x} = \lim_{x \to \infty} \frac{e^x}{2} = \infty$

$\lim_{x \to \infty} e^x = \infty$

$\lim_{x \to \infty} x^2 = \infty$

$\therefore$ L'HOP APPLIES

$\lim_{x \to \infty} e^x = \infty$

$\lim_{x \to \infty} 2x = \infty$

$\therefore$ L'HOP APPLIES