Notes on FTC Part 2 and AROC vs Average Value

We know from part 1 of the fundamental theorem that:

$$\int_a^b f'(x)dx = f(b) - f(a)$$

Part 2 states that

$$\boxed{\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(t)dt\right] = f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)}$$

Let's Prove it:

$$F(x) = \int_{\sin x}^{\ln x} (t^2)dt \qquad \text{Find } F'(x)$$

First, evaluate the integral:

$$F = \frac{t^3}{3}\Big|_{\sin x}^{\ln x} = \frac{(\ln x)^3}{3} - \frac{(\sin x)^3}{3}$$

Now derive.

$$F' = \frac{3(\ln x)^2}{3}\cdot\frac{1}{x} - \frac{3(\sin x)^2}{3}\cdot\cos x = (\ln x)^2\cdot\frac{1}{x} - \sin^2 x\cdot\cos x$$

It would have been much faster to find $F'(x)$ using the 2nd fundamental theorem instead of doing all that work.

Just plug in the limits, and multiply by the derivative of what you plug in. Subtract top – bottom. If either of the limits of integration is a constant, don't forget that the derivative of a constant is 0.

Examples

$$\frac{d}{dx}\int_x^{\pi} \sin t dt =$$

$$\sin\pi\cdot 0 - \sin x\cdot 1$$

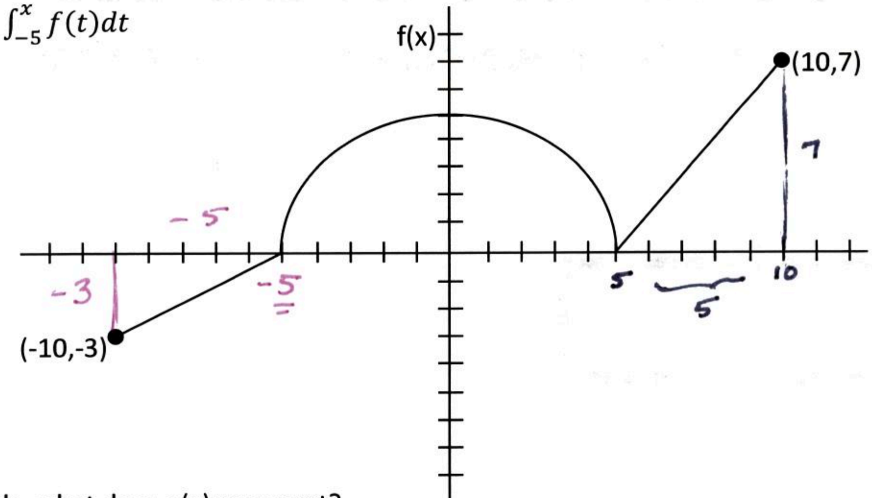
$$= -\sin x$$

$$\frac{d}{dx}\int_1^{x^2} \sqrt{t^3 - t}dt =$$

$$\sqrt{(x^2)^3 - x^2}\cdot 2x - \sqrt{1-1}\cdot 0$$

$$= 2x\sqrt{x^6 - x^2}$$

Given that the following function, f(x), is comprised of two line segments and a semicircle, answer the following questions. $g(x) = \int_{-5}^{x} f(t)dt$



1. In your own words, what does g(x)represent?

The signed area between f(x) & the x-axis from $t = -5$ to $t = x$

2 What is g'(x)? $f(x)$ 2nd FTC $g' = f(x)\cdot 1 - f(-5)\cdot 0 = \underline{\underline{f(x)}}$

3. In your own words, what does g''(x)represent?

$f'(x)$, or the slope of $f(x)$

4. Find g(-5) = 0 $\int_{-5}^{-5} f(t)dt$

5. Find g(5)= $\frac{25\pi}{2}$ $\int_{-5}^{5} f(t)dt$

6. Find g(-10) = 7.5 $\int_{-5}^{-10} f(t)dt$

7. Find g(10) = $\frac{25\pi + 35}{2}$ $\int_{-5}^{10} f(t)dt$

8. Find g'(-10) = -3 $g'(-10) = f(-10)$

9. Find g'(0) = 5 $g'(0) = f(0)$

10. Find g''(0) = 0 $g''(0) = f'(0)$

11. Find g''(6) = $7/5$ $g''(6) = f'(6)$

12. Find g''(-5) = DNE (Sharp turn) $g''(-5) = f'(-5)$

13. On what intervals would g(x) be decreasing? Justify your conclusions.

g is decreasing when $g' = f \leq 0$; ∴ g is decreasing on $[-10, -5]$

14. Find the x-values of any relative extrema for g(x). Justify your conclusions.

g has a relative max when $g' = f$ changes from + to −, ∴ no rel max

g has a relative min when $g' = f$ changes

15. On what intervals is g(x) concave up? Justify your conclusions.

g is concave up when $g' = f$ is increasing. ∴ g is concave up on $(-10,-3) \cup (-3,0) \cup (5,10)$

16. Find the x-values of any points of inflection for g(x). Justify your conclusions.

g has PoI when $g' = f$ has a relative extrema ∴ @ $x = 0, 5$

17. Find the range of g(x). Justify your conclusions. $\left[0, \frac{25\pi + 35}{2}\right]$

↓
Absolute Min to Absolute Max

x	g(x)
-10	7.5
-5	0
5	$25\pi/2$
10	$25\pi + 35$

What is the difference between AROC and Average Value?

The average value of a function is $\frac{1}{b-a}\int_a^b f(x)dx$

AROC is the average rate of change of a function and is $\frac{f(b)-f(a)}{b-a}$.

You need to be very careful that you use these formulas correctly, especially when working velocity.

$\frac{1}{b-a}\int_a^b v(t)dt$ is the average velocity of a function, and using FTC and the fact that $v(t)=x'(t)$

$$\frac{1}{b-a}\int_a^b v(t)dt=\frac{1}{b-a}\int_a^b x'(t)dt=\frac{x(b)-x(a)}{b-a}$$

Example: The velocity of a particle is given by $v(t)=t^2-3t$.

Find the average velocity of the particle over the time interval [1,4].

$$\frac{1}{4-1}\int_1^4 v(t)dt = \frac{\left.\frac{t^3}{3}-\frac{3t^2}{2}\right|_1^4}{3} = \frac{\left(\frac{64}{3}-\frac{48}{2}\right)-\left(\frac{1}{3}-\frac{3}{2}\right)}{3} = \frac{\frac{42}{2}-\frac{45}{2}}{3}$$

$$= -\frac{3}{2}\cdot\frac{1}{3} = -\frac{1}{2}$$

Find the average acceleration of the particle over the time interval [1,4].

OPTION 1

$a(t)=v'(t)=2t-3$

$$\frac{1}{3}\int_1^4 (2t-3)dt = \left.\frac{t^2-3t}{3}\right|_1^4$$

$$\frac{(16-12)-(1-3)}{3} = 2$$

OPTION 2

$$\frac{v(4)-v(1)}{4-1} = \frac{(16-12)-(1-3)}{3}$$

$$=\frac{4+2}{3}$$

$$=2$$

AP Calculus
Worksheet: Definite and Indefinite Integrals Review

1. Suppose that f and g are continuous functions and that

$\int_1^2 f(x)dx = -4$ $\int_1^5 f(x)dx = 6$ $\int_1^5 g(x)dx = 8$

Use the properties of definite integrals to find each integral.

(a) $\int_2^2 g(x)dx$

$= 0$

$\Delta x = 0$

(b) $\int_5^1 g(x)dx$

$-\int_1^5 g(x)\,dx$

$= -8$

(c) $\int_1^2 3f(x)dx$

$3\int_1^2 f(x)\,dx$

$= 3(-4) = -12$

(d) $\int_2^5 f(x)dx$

$= \int_1^5 f(x)\,dx - \int_1^2 f(x)\,dx$

$= 6-(-4) = 10$

(e) $\int_1^5 [f(x) + g(x)]dx$

$= \int_1^5 f(x)\,dx + \int_1^5 g(x)\,dx$

$= 6+8 = 14$

(f) $\int_1^5 [4f(x) - g(x)]dx$

$= 4\int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx$

$= 4\cdot 6 - 8 = 16$

2. Suppose that f and h are continuous functions such that

$\int_1^9 f(x)dx = -1$ $\int_7^9 f(x)dx = 5$ $\int_7^9 h(x)dx = 4$

Use the properties of definite integrals to find each integral.

(a) $\int_1^9 -2f(x)dx$

$-2\int_1^9 f(x)\,dx$

$-2(-1) = 2$

(b) $\int_7^9 [f(x) + h(x)]dx$

$\int_7^9 f(x)\,dx + \int_7^9 h(x)\,dx$

$= 5+4 = 9$

(c) $\int_7^9 [2f(x) - 3h(x)]dx$

$2\int_7^9 f(x)\,dx - 3\int_7^9 h(x)\,dx$

$= 2(5) - 3(4) = -2$

(d) $\int_9^1 f(x)dx$

$-\int_1^9 f(x)\,dx$

$= -(-1) = 1$

(e) $\int_1^7 f(x)dx$

$\int_1^9 f(x)\,dx - \int_7^9 f(x)\,dx$

$= -1-5 = -6$

(f) $\int_9^7 [h(x) - f(x)]dx$

$-\left[\int_7^9 h(x)\,dx - \int_7^9 f(x)\,dx\right]$

$= -[4-5] = 1$

3. Evaluate each integral below.

(a) $\int_3^1 7dx$

$7 \cdot -2$

$= -14$

(b) $\int_0^2 5x\,dx = \frac{5x^2}{2}\Big|_0^2$

$5(2) - 0$

$= 10$

(c) $\int_3^5 \frac{x}{8}dx = \frac{x^2}{16}\Big|_3^5$

$\frac{25}{16} - \frac{9}{16} = \frac{7}{16}$

(d) $\int_0^2 (2t-3)dt$

$(t^2 - 3t)\Big|_0^2$

$(4-6) - 0$

$= -2$

(e) $\int_0^{\sqrt{2}} (t-\sqrt{2})dt$

$\left(\frac{t^2}{2} - \sqrt{2}\,t\right)\Big|_0^{\sqrt{2}}$

$(1-2) - 0$

$= -1$

(f) $\int_2^1 \left(1+\frac{z}{2}\right)dz$

$\left(z + \frac{z^2}{4}\right)\Big|_2^1$

$(1+\frac{1}{4}) - (2+1)$

$= \frac{5}{4} - \frac{12}{4} = \frac{-7}{4}$

(g) $\int_{-1}^1 (x^3 - x)dx$

$\left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Big|_{-1}^1$

$(\frac{1}{4} - \frac{1}{2}) - (\frac{1}{4} - \frac{1}{2})$

$= 0$

(h) $\int_{-1/2}^{1/2} (x^2)dx$

$\frac{x^3}{3}\Big|_{-\frac{1}{2}}^{\frac{1}{2}}$

$(\frac{1}{24}) - (-\frac{1}{24})$

$= \frac{1}{12}$

(i) $\int_1^3 \frac{1}{x}\,dx$

$\ln|x|\Big|_1^3$

$\ln 3 - \cancel{\ln 1}^{0}$

$= \ln 3$

(j) $\int_2^{-1} (3x+1)dx$

$\left(\frac{3x^2}{2} + x\right)\Big|_2^{-1}$

$(\frac{3}{2} - 1) - (6+2) = -7\frac{1}{2}$

(k) $\int_0^{\pi} (\cos x)dx$

$\sin x\Big|_0^{\pi}$

$\sin\pi - \sin 0$

0

(l) $\int_0^{2\pi} (\sin x)\,dx$

$-\cos x\Big|_0^{2\pi}$

$-\cos(2\pi) - (-\cos 0)$

$-1 + 1 = 0$

4. Find dy/dx. -2^{nd} FTC

(a) $y = \int_0^x \sqrt{1+t^2}\,dt$

$y' = \sqrt{1+x^2} \cdot 1 - 0$

$= \sqrt{1+x^2}$

(b) $y = \int_x^1 \frac{1}{t}\,dt$

$y' = \frac{1}{1} \cdot 0 - \frac{1}{x} \cdot 1$

$= -\frac{1}{x}$

(c) $y = \int_0^{\sqrt{x}} \sin(t^2)\,dt$

$y' = \sin x \cdot \frac{1}{2}x^{-1/2} - \sin 0 \cdot 0$

$= \frac{\sqrt{x}\,\sin x}{2}$

(d) $\int_0^{2x} \cos t\,dt$

$\cos(2x) \cdot 2 - \cos 0 \cdot 0$

$2\cos(2x)$

(e) $y = \int_{x^2}^1 (t^2-1)^2 dt$

$(1-1)^2 \cdot 0 - (x^4-1)^2 \cdot 2x$

$= -2x(x^4-1)^2$

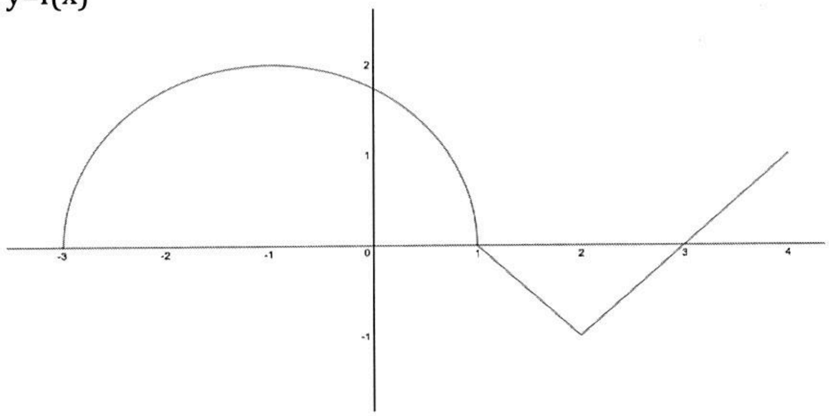
(f) $\int_x^3 \frac{1}{2+e}dt$

$\frac{1}{2+e} \cdot 0 - \frac{1}{2+e} \cdot 1$

$= -\frac{1}{2+e}$

5. The graph of a function f consists of a semicircle and two line segments as shown below.

y=f(x)



Let $g(x) = \int_1^x f(t)dt$

(a) Find g(1)

$g(1) = \int_1^1 f(t)dt$

$= 0$

$(\Delta x = 0)$

(b) Find g(3)

$g(3) = \int_1^3 f(t)dt$

$= \frac{-2 \cdot 1}{2} = -1$

(c) Find g(-1)

$g(-1) = \int_1^{-1} f(t)dt$

$= -\frac{4\pi}{4}$

$= -\pi$

(d) Find all values of x on the open interval (-3,4) at which g has a local minimum.

g has a local min when $g' = f$ changes from − to +

∴ g has a local min @ $x = 3$.

(e) Write an equation for the line tangent to the graph of g at x=-1.

$y - g(-1) = g'(-1)(x+1)$

$y + \pi = 2(x+1)$

$g(-1) = -\pi$

$g'(-1) = f(-1) = 2$

(f) Find the x-coordinate of each point of inflection of the graph of g on the open interval (-3,4).

g has PoI when $g'' = f'$ has a relative extrema

∴ g has PoI @ $x = -1, 2$.

(g) Find the range of g.

x	g(x)
−3	-2π
1	0
3	−1
4	$-\frac{1}{2}$

Range of $g(x) = [-2\pi, 0]$