Key to WKS: Integrating with Inverse Trig

1. $\int \frac{dx/5}{\frac{\sqrt{25-x^2}}{25 \quad 25}} = \frac{1}{5}\int \frac{dx}{\sqrt{1-\left(\frac{x}{5}\right)^2}}$

$u = \frac{x}{5}$

$\frac{du}{dx} = \frac{1}{5}$

$dx = 5du$

$\frac{1}{5}\cdot 5 \int \frac{du}{\sqrt{1-u^2}}$

$= \sin^{-1}\left(\frac{x}{5}\right) + C$

2. $\int \frac{dx/49}{\frac{49+x^2}{49 \quad 49}} = \frac{1}{49}\int \frac{dx}{1+\left(\frac{x}{7}\right)^2}$

$u = \frac{x}{7}$

$\frac{du}{dx} = \frac{1}{7}$

$dx = 7du$

$\frac{1}{49}\cdot 7 \int \frac{du}{1+u^2}$

$\frac{1}{7}\tan^{-1}(u) + C$

$= \frac{1}{7}\tan^{-1}\left(\frac{x}{7}\right) + C$

3. $\int \frac{x}{49+x^2}dx = \frac{1}{2}\int \frac{1}{u}du$

$u = 49 + x^2$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$= \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln(49+x^2) + C$

4. $\int \frac{x^2}{49+x^2}dx$

$$\begin{array}{r} 1 - \frac{49}{49+x^2} \\ x^2+0x+49 \overline{)\, x^2+0x+0} \\ -(x^2+0x+49) \\ \hline -49 \end{array}$$

$\int 1dx - 49\int \frac{1/49}{\frac{49+x^2}{49 \quad 49}}dx$

$x - \int \frac{1}{1+\left(\frac{x}{7}\right)^2}dx$

$= x - 7\tan^{-1}\left(\frac{x}{7}\right) + C$

5. $\int \frac{dx/3}{x\sqrt{\frac{x^2-9}{9 \quad 9}}} = \frac{1}{3}\int \frac{dx}{x\sqrt{\left(\frac{x}{3}\right)^2-1}}$

$u = \frac{x}{3}$

$\frac{du}{dx} = \frac{1}{3}$

$dx = 3du$

$x = 3u$

$= \frac{1}{3}\int \frac{\cancel{3}du}{\cancel{3}u\sqrt{u^2-1}}$

$= \frac{1}{3}\sec^{-1}u + C$

$= \frac{1}{3}\sec^{-1}\left(\frac{x}{3}\right) + C$

6. $\int_0^2 \frac{dx/4}{\frac{4+x^2}{4 \quad 4}} = \frac{1}{4}\int_0^2 \frac{dx}{1+\left(\frac{x}{2}\right)^2}$

$u = \frac{x}{2}$

$\frac{du}{dx} = \frac{1}{2}$

$dx = 2du$

$= \frac{1}{2}\int_0^1 \frac{du}{1+u^2}$

$= \frac{1}{2}\tan^{-1}u \Big|_0^1$

$= \frac{1}{2}\tan^{-1}(1) - \frac{1}{2}\tan^{-1}(0)$

$= \frac{1}{2}\left(\frac{\pi}{4}\right) - \frac{1}{2}(0)$

$= \frac{\pi}{8}$

7. $\int_{\frac{1}{2}}^1 \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x)\Big|_{\frac{1}{2}}^1$

$\sin^{-1}(1) - \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2} - \frac{\pi}{6}$

$= \frac{\pi}{3}$

8. $\int \frac{3}{x^2+2x+10}dx = 3\int \frac{1}{\left(x^2+2x+1\right)+10-1}dx$

$= 3\int \frac{1}{(x+1)^2+9}dx$

$= 3\int \frac{\frac{1}{9}}{\frac{(x+1)^2}{9}+\frac{9}{9}}dx$

$u = \frac{x+1}{3} = \frac{1}{3}x+\frac{1}{3}$
$\frac{du}{dx} = \frac{1}{3}$
$dx = 3du$

$= \frac{1}{3}\int \frac{1}{\left(\frac{x+1}{3}\right)^2+1}dx$

$= \int \frac{1}{u^2+1}du$

$= \tan^{-1}(u)+C$

$= \tan^{-1}\left(\frac{x+1}{3}\right)+C$

9. $\int_0^3 e^{-2x}\,dx = -\frac{1}{2}\int_0^{-6} e^u\,du$

$u = -2x$ $\quad = -\frac{1}{2}e^u\Big|_0^{-6}$

$\frac{du}{dx} = -2$ $\quad = -\frac{1}{2}e^{-6} + \frac{1}{2}\cancel{e^0}^{\,1}$

$dx = \frac{du}{-2}$

$= -\frac{1}{2}e^{-6}+\frac{1}{2}$

10. $\int_{\frac{\pi}{8}}^{\frac{\pi}{6}} \cos(2x)\,dx = 2\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos u\,du$

$u = 2x$ $\quad = 2\sin u\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$\frac{du}{dx} = 2$

$dx = 2dx$ $\quad = 2\sin\left(\frac{\pi}{3}\right) - 2\sin\frac{\pi}{4}$

$= 2\left(\frac{\sqrt{3}}{2}\right) - 2\left(\frac{\sqrt{2}}{2}\right)$

$= \sqrt{3}-\sqrt{2}$

11. $\int_{\frac{\pi}{12}}^{\pi/9} \sin(3x)\cos^2(3x)\,dx$

$u = \cos(3x)$ $\quad = -\frac{1}{3}\int_{\frac{\sqrt{2}}{2}}^{\frac{1}{2}} u^2\,du$

$\frac{du}{dx} = -3\sin 3x$

$dx = \frac{du}{-3\sin 3x}$ $\quad = -\frac{1}{3}\cdot\frac{u^3}{3}\Big|_{\frac{\sqrt{2}}{2}}^{\frac{1}{2}}$

$= -\frac{1}{9}\left(\frac{1}{2}\right)^3 + \frac{1}{9}\left(\frac{\sqrt{2}}{2}\right)^3$

$= -\frac{1}{72}+\frac{\sqrt{8}}{72}$

$= \frac{\sqrt{8}-1}{72}$

12. $\int_0^4 \frac{x}{\sqrt{9+x^2}}\,dx = \frac{1}{2}\int_9^{25} u^{-1/2}\,du$

$u = 9+x^2$ $\quad = \frac{1}{2}\cdot 2\cdot u^{1/2}\Big|_9^{25}$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$ $\quad = 5-3$

$= 2$

3. $\int_0^4 \frac{5}{3x+1}\,dx = \frac{5}{3}\int_1^{13} \frac{1}{u}\,du$

$u = 3x+1$ $\quad \frac{5}{3}\ln|u|\Big|_1^{13}$

$\frac{du}{dx} = 3$

$dx = \frac{du}{3}$ $\quad \frac{5}{3}\ln 13 - \cancel{\frac{5}{3}\ln 1}^{\,0}$

$= \frac{5}{3}\ln 13$

14. $\int \frac{x}{x^2}dx + \int \frac{2x^3}{x^2}dx + \int \frac{1}{x^2}dx$

$\int \frac{1}{x}dx + \int 2x\,dx + \int x^{-2}dx$

$= \ln|x| + x^2 - \frac{1}{x} + C$

16. $\int_e^{e^3} \frac{(\ln x + 2)^3}{x}dx = \int_3^5 u^3\,du$

$u = \ln x + 2$

$\frac{du}{dx} = \frac{1}{x}$

$dx = x \cdot du$

$= \frac{u^4}{4}\Big|_3^5$

$= \frac{625}{4} - \frac{81}{4}$

$= 136$

15. $\int_0^2 \frac{e^{3x}}{1+e^{3x}}dx = \frac{1}{3}\int_2^{1+e^6} \frac{1}{u}\,du$

$u = 1 + e^{3x}$

$\frac{du}{dx} = 3e^{3x}$

$dx = \frac{du}{3e^{3x}}$

$= \frac{1}{3}\ln|u|\Big|_2^{1+e^6}$

$= \frac{1}{3}\ln(1+e^6) - \frac{1}{3}\ln 2$

$= \frac{1}{3}\ln\left(\frac{1+e^6}{2}\right)$

17.

a) Avg rate: $\frac{52-60}{5-2}$ °C / min

$\frac{-8}{3}$ °C / min

b) $\frac{1}{10}\int_0^{10} H(t)\,dt$ is the Average Temperature of the tea during the first 10 minutes in degrees Celsius.

$\frac{1}{10}\left(\frac{66+60}{2}\cdot 2 + \frac{60+52}{2}\cdot 3 + \frac{52+44}{2}\cdot 4 + \frac{44+43}{2}\cdot 1\right)$

$= 52.95$°C

c) $\int_0^{10} H'(t)\,dt = H(10) - H(0)$

$= 43 - 66 = -23$°C

The tea cooled down 23°C during the first 10 minutes. The expression represents the change in temperature of the tea during the first 10 min.

d) $100 + \int_0^{10} B'(t)\,dt$ = the biscuits temperature when $t = 10$

$B(10) = 34.182$°C

$H(10) - B(10) = 43 - 34.182 = 8.817$°C

The biscuits are 8.817°C cooler than the tea.