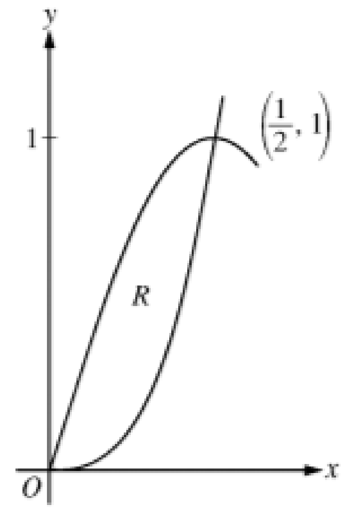


Let R be the region in the first quadrant enclosed by the graphs of $f(x) = 8x^3$ and $g(x) = \sin(\pi x)$, as shown in the figure above.

(a) Write an equation for the line tangent to the graph of f at $x = \frac{1}{2}$.

(b) Find the area of R.

(c) Write, but do not evaluate, an integral expression for the volume of the solid generated when R is rotated about the horizontal line $y = 1$.

(a) $f\left(\frac{1}{2}\right) = 1$

$f'(x) = 24x^2$, so $f'\left(\frac{1}{2}\right) = 6$

An equation for the tangent line is $y = 1 + 6\left(x - \frac{1}{2}\right)$.

$2 : \begin{cases} 1 : f'\left(\frac{1}{2}\right) \\ 1 : \text{answer} \end{cases}$

(b) Area $= \int_0^{1/2} (g(x) - f(x))\,dx$

$= \int_0^{1/2} \left(\sin(\pi x) - 8x^3\right) dx$

$= \left[-\frac{1}{\pi}\cos(\pi x) - 2x^4\right]_{x=0}^{x=1/2}$

$= -\frac{1}{8} + \frac{1}{\pi}$

$4 : \begin{cases} 1 : \text{integrand} \\ 2 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(c) $\pi\int_0^{1/2} \left((1 - f(x))^2 - (1 - g(x))^2\right) dx$

$= \pi\int_0^{1/2} \left(\left(1 - 8x^3\right)^2 - (1 - \sin(\pi x))^2\right) dx$

$3 : \begin{cases} 1 : \text{limits and constant} \\ 2 : \text{integrand} \end{cases}$

2.

Let f and g be the functions given by $f(x) = e^x$ and $g(x) = \ln x$.

(a) Find the area of the region enclosed by the graphs of f and g between $x = \frac{1}{2}$ and $x = 1$.

(b) Find the volume of the solid generated when the region enclosed by the graphs of f and g between $x = \frac{1}{2}$ and $x = 1$ is revolved about the line $y = 4$.

(c) Let h be the function given by $h(x) = f(x) - g(x)$. Find the absolute minimum value of $h(x)$ on the closed interval $\frac{1}{2} \le x \le 1$, and find the absolute maximum value of $h(x)$ on the closed interval $\frac{1}{2} \le x \le 1$. Show the analysis that leads to your answers.

(a) Area $= \int_{1/2}^{1} (e^x - \ln x)\,dx = 1.222$ or 1.223

$2 \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

(b) Volume $= \pi \int_{1/2}^{1} \left((4 - \ln x)^2 - (4 - e^x)^2\right) dx$

$= 7.515\pi$ or 23.609

$4 \begin{cases} 1: \text{limits and constant} \\ 2: \text{integrand} \\ \quad <-1> \text{ each error} \\ \quad \text{Note: } 0/2 \text{ if not of the form} \\ \quad k\int_a^b \left(R(x)^2 - r(x)^2\right) dx \\ 1: \text{answer} \end{cases}$

(c) $h'(x) = f'(x) - g'(x) = e^x - \frac{1}{x} = 0$

$x = 0.567143$

Absolute minimum value and absolute maximum value occur at the critical point or at the endpoints.

$h(0.567143) = 2.330$
$h(0.5) = 2.3418$
$h(1) = 2.718$

This is a candidates test.

The absolute minimum is 2.330.
The absolute maximum is 2.718.

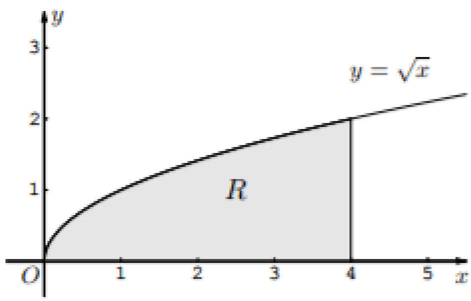
$3 \begin{cases} 1: \text{considers } h'(x) = 0 \\ 1: \text{identifies critical point} \\ \quad \text{and endpoints as candidates} \\ 1: \text{answers} \end{cases}$

Note: Errors in computation come off the third point.

3.

1. Let R be the region bounded by the x–axis, the graph of $y = \sqrt{x}$, and the line $x = 4$.
   (a) Find the area of the region R.
   (b) Find the value of h such that the vertical line $x = h$ divides the region R into two regions of equal area.
   (c) Find the volume of the solid generated when R is revolved about the x–axis.
   (d) The vertical line $x = k$ divides the region R into two regions such that when these two regions are revolved about the x–axis, they generate solids with equal volumes. Find the value of k.

(a)



$$A = \int_0^4 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_0^4 = \frac{16}{3} \quad \text{or} \quad 5.333$$

$2 \begin{cases} 1: & A = \int_0^4 \sqrt{x}\,dx \\ 1: & \text{answer} \end{cases}$

(b) $\int_0^h \sqrt{x}\,dx = \frac{8}{3}$ –or– $\int_0^h \sqrt{x}\,dx = \int_h^4 \sqrt{x}\,dx$

$\frac{2}{3}h^{3/2} = \frac{8}{3}$ –or– $\frac{2}{3}h^{3/2} = \frac{16}{3} - \frac{2}{3}h^{3/2}$

$h = \sqrt[3]{16}$ or 2.520 or 2.519

$2 \begin{cases} 1: & \text{equation in } h \\ 1: & \text{answer} \end{cases}$

(c) $V = \pi \int_0^4 (\sqrt{x})^2\,dx = \pi \frac{x^2}{2}\Big|_0^4 = 8\pi$

or 25.133 or 25.132

$3 \begin{cases} 1: & \text{limits and constant} \\ 1: & \text{integrand} \\ 1: & \text{answer} \end{cases}$

(d) $\pi \int_0^k (\sqrt{x})^2\,dx = 4\pi$ –or– $\pi \int_0^k (\sqrt{x})^2\,dx = \pi \int_k^4 (\sqrt{x})^2\,dx$

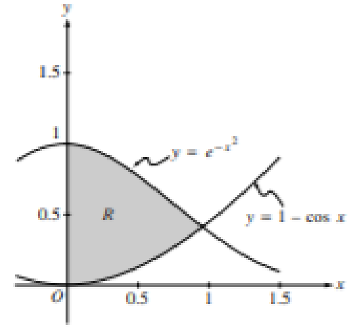
$\pi \frac{k^2}{2} = 4\pi$ –or– $\pi \frac{k^2}{2} = 8\pi - \pi \frac{k^2}{2}$

$k = \sqrt{8}$ or 2.828

$2 \begin{cases} 1: & \text{equation in } k \\ 1: & \text{answer} \end{cases}$

4.

Let R be the shaded region in the first quadrant enclosed by the graphs of $y = e^{-x^2}$, $y = 1 - \cos x$, and the y-axis, as shown in the figure above.



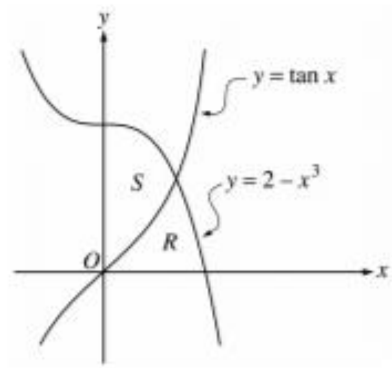
(a) Find the area of the region R.

(b) Find the volume of the solid generated when the region R is revolved about the x-axis.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

Region R $e^{-x^2} = 1 - \cos x$ at $x = 0.941944 = A$	1 : Correct limits in an integral in (a), (b), or (c).
(a) Area $= \int_0^A (e^{-x^2} - (1 - \cos x))\,dx$ $= 0.590$ or 0.591	2 { 1 : integrand 1 : answer
(b) Volume $= \pi \int_0^A \left(\left(e^{-x^2}\right)^2 - (1 - \cos x)^2 \right) dx$ $= 0.55596\pi = 1.746$ or 1.747	3 { 2 : integrand and constant < −1> each error 1 : answer
(c) Volume $= \int_0^A \left(e^{-x^2} - (1 - \cos x)\right)^2 dx$ $= 0.461$	3 { 2 : integrand < −1> each error Note: 0/2 if not of the form $k \int_c^d (f(x) - g(x))^2\,dx$ 1 : answer

5.



Let R and S be the regions in the first quadrant shown in the figure above. The region R is bounded by the x-axis and the graphs of $y = 2 - x^3$ and $y = \tan x$. The region S is bounded by the y-axis and the graphs of $y = 2 - x^3$ and $y = \tan x$.

(a) Find the area of R.

(b) Find the area of S.

(c) Find the volume of the solid generated when S is revolved about the x-axis.

Point of intersection

$2 - x^3 = \tan x$ at $(A, B) = (0.902155, 1.265751)$

(a) Area $R = \int_0^A \tan x\, dx + \int_A^{\sqrt[3]{2}} (2 - x^3)\, dx = 0.729$

or

Area $R = \int_0^B \left((2-y)^{1/3} - \tan^{-1} y\right) dy = 0.729$

or

Area $R = \int_0^{\sqrt[3]{2}} (2 - x^3)\, dx - \int_0^A (2 - x^3 - \tan x)\, dx = 0.729$

3: { 1 : limits; 1 : integrand; 1 : answer }

(b) Area $S = \int_0^A (2 - x^3 - \tan x)\, dx = 1.160$ or 1.161

or

Area $S = \int_0^B \tan^{-1} y\, dy + \int_B^2 (2-y)^{1/3}\, dy = 1.160$ or 1.161

or

Area S

$= \int_0^2 (2-y)^{1/3}\, dy - \int_0^B \left((2-y)^{1/3} - \tan^{-1} y\right) dy$

$= 1.160$ or 1.161

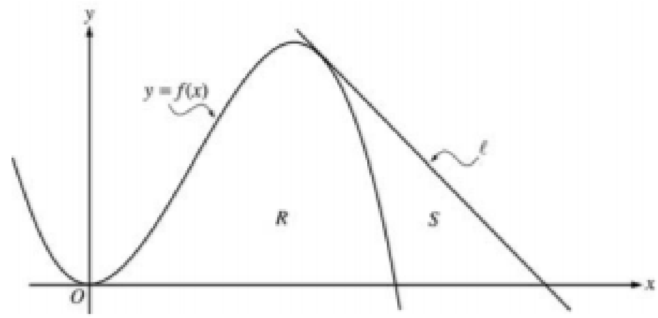
3: { 1 : limits; 1 : integrand; 1 : answer }

(c) Volume $= \pi \int_0^A \left((2 - x^3)^2 - \tan^2 x\right) dx$

$= 2.652\pi$ or 8.331 or 8.332

3: { 1 : limits and constant; 1 : integrand; 1 : answer }

6.



Let f be the function given by $f(x) = 4x^2 - x^3$, and let ℓ be the line $y = 18 - 3x$, where ℓ is tangent to the graph of f. Let R be the region bounded by the graph of f and the x-axis, and let S be the region bounded by the graph of f, the line ℓ, and the x-axis, as shown above.

(a) Show that ℓ is tangent to the graph of $y = f(x)$ at the point $x = 3$.

(b) Find the area of S.

(c) Find the volume of the solid generated when R is revolved about the x-axis.

(a) $f'(x) = 8x - 3x^2$; $f'(3) = 24 - 27 = -3$

$f(3) = 36 - 27 = 9$

Tangent line at $x = 3$ is

$y = -3(x-3) + 9 = -3x + 18$,

which is the equation of line ℓ.

2 : { 1 : finds $f'(3)$ and $f(3)$
1 : { finds equation of tangent line
or
shows (3,9) is on both the graph of f and line ℓ } }

(b) $f(x) = 0$ at $x = 4$

The line intersects the x-axis at $x = 6$.

$$\text{Area} = \frac{1}{2}(3)(9) - \int_3^4 (4x^2 - x^3)\,dx$$
$$= 7.916 \text{ or } 7.917$$

OR

$$\text{Area} = \int_3^4 \left((18 - 3x) - (4x^2 - x^3)\right)dx + \frac{1}{2}(2)(18 - 12)$$
$$= 7.916 \text{ or } 7.917$$

4 : { 2 : integral for non-triangular region
1 : limits
1 : integrand
1 : area of triangular region
1 : answer }

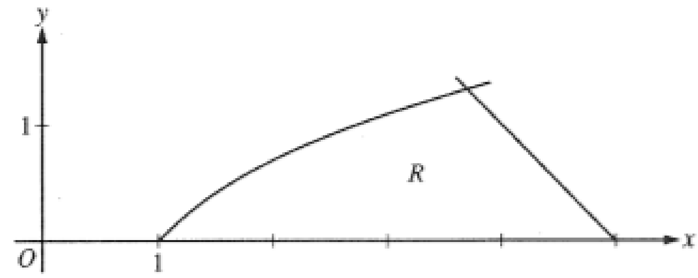
(c) $$\text{Volume} = \pi\int_0^4 (4x^2 - x^3)^2\,dx$$
$$= 156.038\pi \text{ or } 490.208$$

3 : { 1 : limits and constant
1 : integrand
1 : answer }

7.

Question 2

Let R be the region in the first quadrant bounded by the x-axis and the graphs of $y = \ln x$ and $y = 5 - x$, as shown in the figure above.



(a) Find the area of R.

(b) Region R is the base of a solid. For the solid, each cross section perpendicular to the x-axis is a square. Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid.

(c) The horizontal line $y = k$ divides R into two regions of equal area. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of k.

$\ln x = 5 - x \;\Rightarrow\; x = 3.69344$

Therefore, the graphs of $y = \ln x$ and $y = 5 - x$ intersect in the first quadrant at the point $(A, B) = (3.69344, 1.30656)$.

(a) $\text{Area} = \int_0^B \left(5 - y - e^y\right)\,dy$
$= 2.986 \text{ (or } 2.985)$

OR

$\text{Area} = \int_1^A \ln x\,dx + \int_A^5 (5 - x)\,dx$
$= 2.986 \text{ (or } 2.985)$

3 : { 1 : integrand; 1 : limits; 1 : answer }

(b) $\text{Volume} = \int_1^A (\ln x)^2\,dx + \int_A^5 (5 - x)^2\,dx$

3 : { 2 : integrands; 1 : expression for total volume }

(c) $\int_0^k \left(5 - y - e^y\right)\,dy = \frac{1}{2} \cdot 2.986 \left(\text{or } \frac{1}{2} \cdot 2.985\right)$

3 : { 1 : integrand; 1 : limits; 1 : equation }