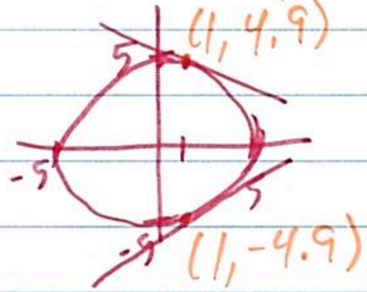
# 3.5 Implicit & Logarithmic Differentiation

Take the equation $x^2+y^2=25$




There are 2 tangent lines when $x=1$

$$x^2+y^2=25$$
$$\cancel{-x^2} \qquad -x^2$$
$$y^2=-x^2+25$$
$$y=\pm\sqrt{-x^2+25}$$

When $x=1$

$y=\pm\sqrt{-(1)^2+25}$

$y=\pm\sqrt{24}$

$y\approx\pm 4.9$

↳ To take a derivative we would need to differentiate 2 messy functions.

Other functions are such a mess that we might not be able to solve for y at all.

Hence: Implicit Differentiation
↳ (not obvious - implied)

$\frac{dy}{dx}$ means derivative of y with respect to x

$\frac{dx}{dx}$ means derivative of x with repect to x

We have been taking the derivative of both sides this whole time.

$$\frac{d}{dx}\left[y = x^2\right]$$

$$1\frac{dy}{dx} = 2x\cancel{\frac{dx}{dx}}$$

$$\frac{dy}{dx} = 2x$$

$$y = x^2$$
$$y' = 2x$$

Back to our circle. Find $\frac{dy}{dx}$

$$\frac{d}{dx}\left[x^2 + y^2 = 25\right]$$

$$2x\cancel{\frac{dx}{dx}} + 2y\frac{dy}{dx} = 0$$

$$-2x \qquad -2x$$

$$\frac{\cancel{2y}\frac{dy}{dx}}{\cancel{2y}} = \frac{-2x}{2y}$$

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

This is the formula for the slope of the tangent line.

At $(1, 4.9)$ the slope is $-\frac{1}{4.9}$

At $(1, -4.9)$ the slope is $\frac{1}{4.9}$

Ex: $3y^2 + 2x - y = 4$ Find $\frac{dy}{dx}$

$$\frac{d}{dx}\left[3y^2 + 2x - y = 4\right]$$

$\cancel{\frac{d\#}{dx}}$

$$6y\frac{dy}{dx} + 2\cancel{\frac{dx}{dx}} - 1\frac{dy}{dx} = 0$$

$$6y\frac{dy}{dx} - 1\frac{dy}{dx} = -2$$

$$\frac{\frac{dy}{dx}(6y-1)}{6y-1} = \frac{-2}{6y-1}$$

$$\boxed{\frac{dy}{dx} = \frac{-2}{6y-1}}$$

$\frac{d}{dx}\left[9x^2 - xy + y^2 = 0\right]$ Find $\frac{dy}{dx}$

$$18x\cancel{\frac{dx}{dx}} - x \cdot 1\frac{dy}{dx} + y \cdot -1\cancel{\frac{dx}{dx}} + 2y\frac{dy}{dx} = 0$$

$$-x\frac{dy}{dx} + 2y\frac{dy}{dx} = -18x + y$$

$$\frac{\frac{dy}{dx}\cancel{(2y-x)}}{\cancel{2y-x}} = \frac{y-18x}{2y-x}$$

$$\boxed{\frac{dy}{dx} = \frac{y-18x}{2y-x}}$$

When do we use implicit differentiation?
When $y$ is not by itself
or it is inconvenient to get
$y$ by itself.

Second Derivatives $\frac{d^2y}{dx^2}$

Does NOT mean anything is squared.

$$\underbrace{\frac{d}{dx}\left[\frac{d}{dx}\left[y=x\right]\right]}_{\frac{d^2y}{dx^2}}$$

kind of explains notation

Back to our circle.

$$\frac{d}{dx}\left[\frac{dy}{dx}=-\frac{x}{y}\right]$$

Let's find $\frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2}=\frac{y(-1)\cancel{\frac{dx}{dx}}-(-x)(1)\frac{dy}{dx}}{y^2}$$

$$\frac{d^2y}{dx^2}=\frac{-y+x\frac{dy}{dx}}{y^2}$$

$$\frac{d^2y}{dx^2}=\frac{-y+x\left(\frac{-x}{y}\right)}{y^2}$$

$$\frac{d^2y}{dx^2}=\frac{-y-\frac{x^2}{y}}{y^2}\cdot\frac{y}{y}$$

$$\boxed{\frac{d^2y}{dx^2}=\frac{-y^2-x^2}{y^3}}$$

$$\frac{-1(x^2+y^2)}{y^3}$$

# Logarithmic Differentiation

Can be used anytime, but it is taylor made for function

$$y = a^x$$

$$\ln y = \ln a^x$$

$$\frac{d}{dx}\left[\ln y = x \ln a\right]$$ Take derivative of both sides

$$\frac{1}{y}\frac{dy}{dx} = \cancel{x \cdot (0)} + \ln a \cdot 1 \cancel{\frac{dx}{dx}}$$

$$(y)\frac{1}{y}\frac{dy}{dx} = \ln a \, (y)$$

$$\frac{dy}{dx} = \ln a \cdot a^x$$

$$\frac{dy}{dx} = a^x \ln a$$

Ex $y = x\sqrt{x^2-4}$ → could be solved using product & chain rule

$$\ln y = \ln\left(x\sqrt{x^2-4}\right)$$

$$\ln y = \ln x + \ln\left(x^2-4\right)^{1/2}$$

$$\frac{d}{dx}\left[\ln y = \ln x + \frac{1}{2}\ln\left(x^2-4\right)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\cancel{\frac{dx}{dx}} + \frac{1}{2}\cdot\frac{2x}{x^2-4}\cancel{\frac{dx}{dx}}$$

$$\cancel{y}\cdot\cancel{\frac{1}{y}}\frac{dy}{dx} = \frac{1}{x} + \frac{x}{x^2-4} \cdot y$$

$$\frac{dy}{dx} = \left(\frac{1}{x} + \frac{x}{x^2-4}\right)\left(x\sqrt{x^2-4}\right)$$

$$y = (2x+1)^{3x}$$

$$\ln y = \ln (2x+1)^{3x} \qquad \frac{du}{u}$$

$$\frac{d}{dx}\left[\ln y = 3x \ln(2x+1)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = 3x \cdot \frac{2}{2x+1}\cancel{\frac{dx}{dx}} + \ln(2x+1) \cdot 3\cancel{\frac{dx}{dx}}$$

$$y \cdot \cancel{\frac{1}{y}}\frac{dy}{dx} = \frac{6x}{2x+1} + 3\ln(2x+1) \cdot y$$

$$\frac{dy}{dx} = \left[\frac{6x}{2x+1} + 3\ln(2x+1)\right](2x+1)^{3x}$$