AP Calculus BC

5.2 Area and 5.3 Riemann Sums and Definite Integrals

Definite Integral

$\int_a^b f(x)dx = \textit{Numerical Value}$ - the actual area under a curve to the x-axis from a to b.

Continuity Implies Integrability

If a function f is continuous on the closed interval [a, b], then f is integrable on [a, b].

Two ways to evaluate a definite integral, at this time –

1) Rectangle methods/Trapezoids – this will usually only give an estimate

2) Use a geometric formula – this will give an exact value if the function is a known shape.

A third way to evaluate definite integrals will be introduced soon – the anitderivative.

Riemann Sum

If $f(x)$ is continuous and non-negative on the interval [a, b], then the area of the region bounded by the graph of f, the x-axis, and the vertical lines x = a and x = b is

$$\text{Area} = \int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left[f(c_i)\Delta x\right]; \qquad \Delta x = \frac{b-a}{n}$$

Right Sum:

$c_i = a + \Delta x\, i$

Left Sum:

$c_i = a + \Delta x\,(i-1)$

Example:

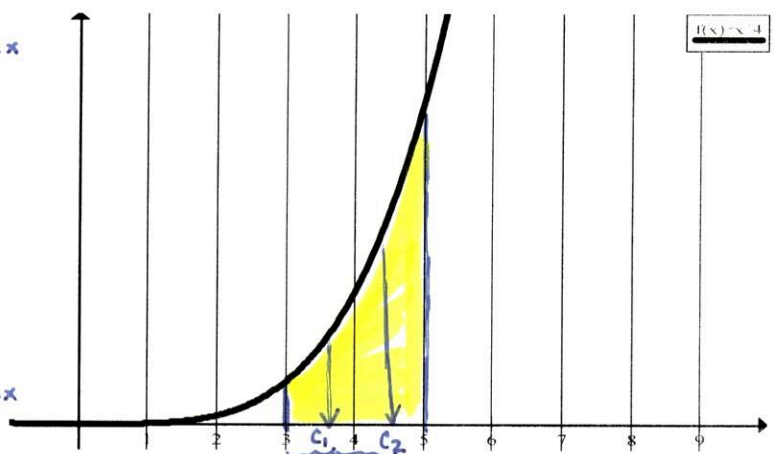
Which of the following limits is equal to $\int_3^5 x^4\, dx$?

a) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(3+\frac{k}{n}\right)^4\frac{1}{n}$ wrong Δx

b) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(3+\frac{k}{n}\right)^4\frac{2}{n}$ missing the 2

c) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(3+\frac{2k}{n}\right)^4\frac{1}{n}$ wrong Δx

d) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(3+\frac{2k}{n}\right)^4\frac{2}{n}$ Correct!



$(3+\Delta x)$ $(3+2\Delta x)$

$\Delta x = \frac{5-3}{n} = \frac{2}{n}$

$c_k = 3 + \frac{2}{n}k$

$f(c_k) = \left(3+\frac{2}{n}k\right)^4$

Practice: Use a right sum to write the limit definition of the integral $\int_2^5 (x^2 - \sin x)dx$.

$\Delta x = \frac{5-2}{n} = \frac{3}{n}$

$c_i = 2 + \frac{3}{n} i$

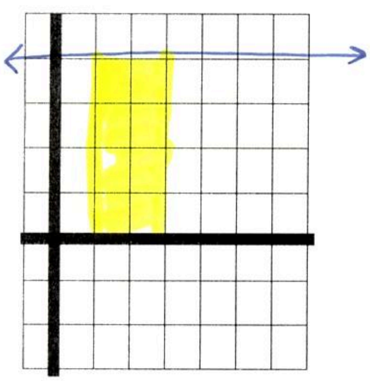
$f(c_i) = (2 + \frac{3}{n} i)^2 - \sin(2 + \frac{3}{n} i)$

$$\int_2^5 (x^2 - \sin x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \left\{ \left[(2 + \tfrac{3}{n} i)^2 - \sin(2 + \tfrac{3}{n} i) \right] \frac{3}{n} \right\}$$

Using Geometry to find the value of an integral:

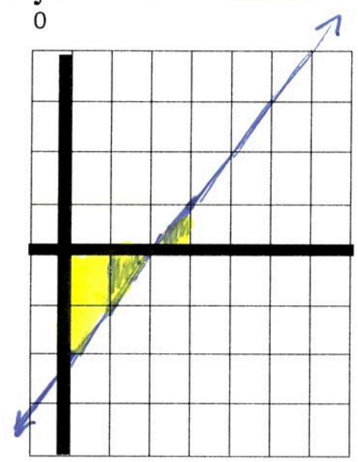
Sketch the region corresponding to each definite integral. Then evaluate each integral using a geometric formula.

1) $\int_1^3 4dx =$ 8

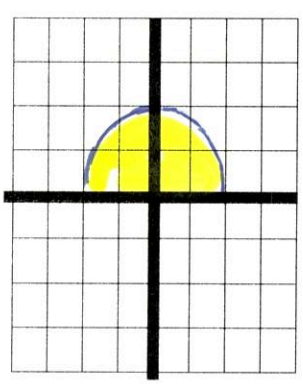


area = 4·2

2) $\int_0^3 (x-2)dx =$ -1.5



3) $\int_{-2}^{2} \sqrt{4-x^2}\,dx =$ 2π



$y = \sqrt{4-x^2}$

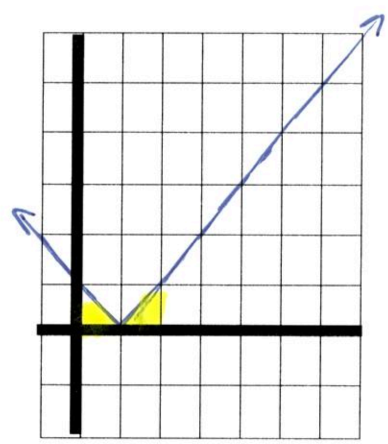
$y^2 = 4 - x^2$

$x^2 + y^2 = 4$

Top half of a circle w/ r = 2

area $= \frac{\pi(2)^2}{2}$

4) $\int_0^2 |x-1| dx =$ 1



area = $\frac{1}{2} \cdot 2$

Properties of Definite Integrals

A) If f is defined at x = a, then $\int_{a}^{a} f(x)dx = 0$

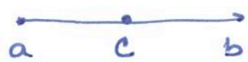
$\Delta x = 0$

B) If f is integrable on [a, b], then $\int_{b}^{a} f(x)dx = -\int_{a}^{b} f(x)dx$

filling emptying

C) If f is integrable on the three closed intervals determined by a, b, and c, then

$$\int_{a}^{b} f(x)dx = \int_{a}^{c} f(x)dx + \int_{c}^{b} f(x)dx$$



D) If f is integrable on the closed interval [a, b] and $\int_{a}^{b} f(x)dx = c$, then $\int_{a}^{b} k \cdot f(x)dx = k \cdot c$, .

* just pull out the constant!

$k \int_a^b f(x)\,dx$

Using Properties to evaluate integrals:

5) Given $\int_{0}^{3} f(x)dx = 4$ and $\int_{3}^{6} f(x)dx = -1$ evaluate:

a) $\int_{0}^{6} f(x)dx = \underline{\underline{3}}$

$= \int_0^3 f(x)\,dx + \int_3^6 f(x)\,dx$

$= 4 + (-1)$

b) $\int_{6}^{3} f(x)dx = \underline{\underline{1}}$

$= -\int_3^6 f(x)\,dx$

$= -(-1)$

c) $\int_{3}^{3} f(x)dx = \underline{\underline{0}}$

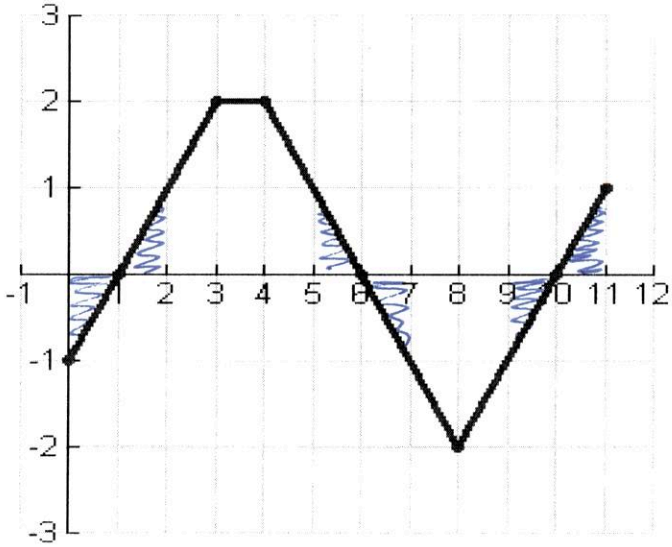
$\Delta x = 0$

d) $\int_{3}^{6} -5f(x)dx = \underline{\underline{5}}$

$= -5 \int_3^6 f(x)\,dx$

$= -5(-1)$

6) The graph of f consists of line segments, as shown in the figure. Evaluate each definite integral by using geometric formulas.



a) $\int_0^1 -f(x)dx$ $= \underline{\underline{\frac{1}{2}}}$

$= -\int_0^1 f(x)\,dx$

$= -\left(-\frac{1}{2}\right)$

b) $\int_3^4 3f(x)dx$ $= \underline{\underline{6}}$

$= 3\int_3^4 f(x)\,dx$

$= 3\,(2)$

c) $\int_0^7 f(x)dx$ $= \underline{\underline{5}}$

$= \int_2^5 f(x)\,dx$

d) $\int_5^{11} f(x)dx$ $= \underline{\underline{-3}}$

$= \int_7^9 f(x)\,dx$

e) $\int_0^{11} f(x)dx$ $= \underline{\underline{2}}$

$= \int_0^7 f(x)dx + \int_7^{11} f(x)dx$

$= 5 + \int_7^9 f(x)dx$

$= 5 - 3$

f) $\int_0^7 [f(x)+2]dx$ $= \underline{\underline{19}}$

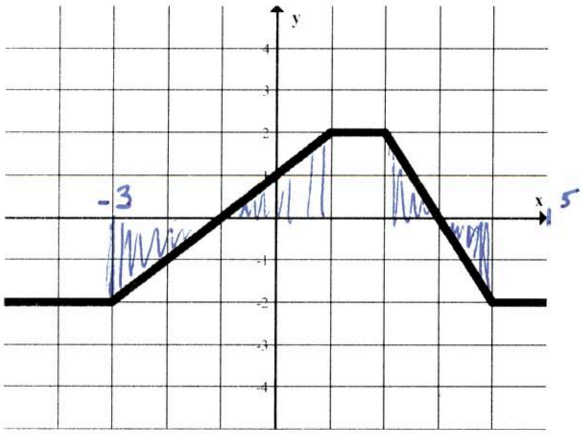
$= \int_0^7 f(x)\,dx + \int_0^7 2\,dx$

$= 5 + 14$

Area – AB Calculus

name *Key*

1.



$A(t) = \int_{-3}^{t} f(x)dx$

a) A(-3) = $\int_{-3}^{-3} f(x)dx = \underline{\underline{0}}$

b) A(-1) = $\int_{-3}^{-1} f(x)dx = \underline{\underline{-2}}$

c) A(1) = $\int_{-3}^{1} f(x)dx = \underline{\underline{0}}$

d) A(5) = $\int_{-3}^{5} f(x)dx = \underline{\underline{0}}$

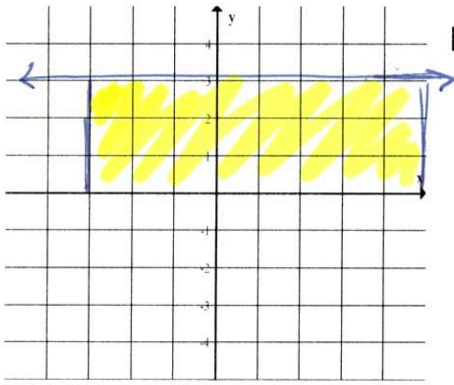
e) A(-5) = $\int_{-3}^{-5} f(x)dx = \underline{\underline{4}}$

2. Evaluate the following integrals by making a graph of the function over the relevant interval. {Use geometry.}

a) $\int_{-3}^{5} 3dx$

$3 \cdot 8$

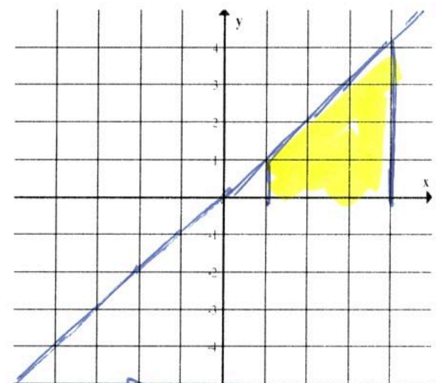
$= \underline{\underline{24}}$



b) $\int_{1}^{4} xdx$

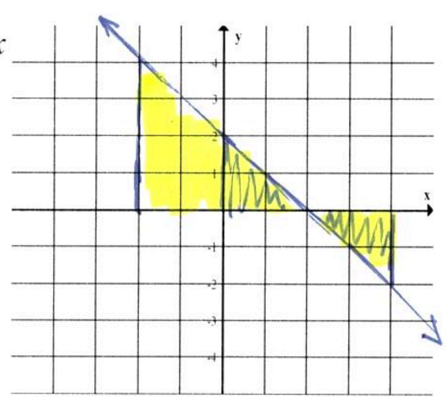
$\frac{1+4}{2} \cdot 3$

$= \underline{\underline{\frac{15}{2}}}$



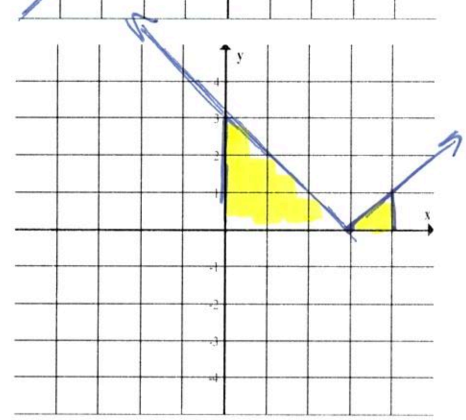
c) $\int_{-2}^{4} (2-x)dx$

$= \underline{\underline{6}}$



d) $\int_{0}^{4} |x-3|\, dx$

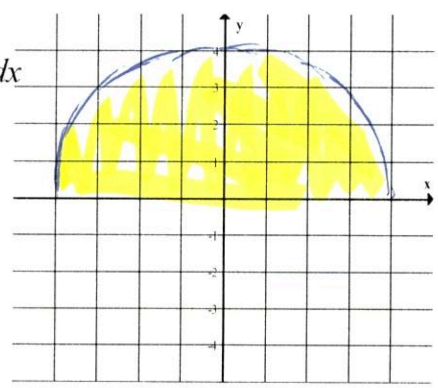
$= \underline{\underline{5}}$



e) $\int_{-4}^{4} \sqrt{16-x^2}\, dx$

$\frac{\pi(4)^2}{2}$

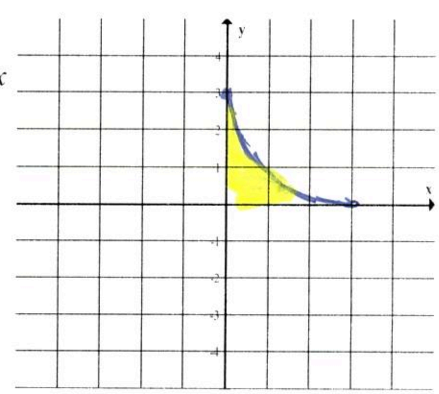
$= \underline{\underline{8\pi}}$



f) $\int_{0}^{3} (3-\sqrt{9-x^2})dx$

up 3 + flip

$9 - \frac{9\pi}{4}$



Calculus Worksheet on Integrals and Area Name ______________________________

1. Which of the following definite integrals is equal to $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{12k}{n}\cos\left(1+\frac{4k}{n}\right)\frac{4}{n}$?

$\Delta x = \frac{4}{n}$

$c_k = 1 + \frac{4}{n}k$

$c_k = 0 + \frac{4}{n}k$

A) $\int_1^5 12\cos x\,dx$

$\lim_{n\to\infty}\sum_{k=1}^{n} 12\cos\left(1+\frac{4}{n}k\right)\frac{4}{n}$

C) $\int_0^4 12\cos(1+x)\,dx$

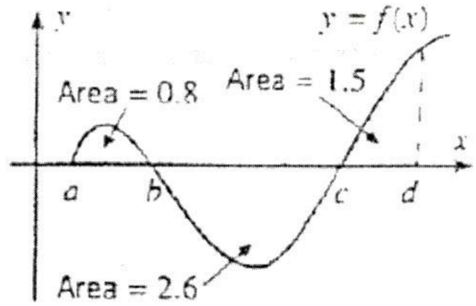
$\lim_{n\to\infty} 12\cos\left(1+\frac{4}{n}k\right)\frac{4}{n}$

B) $\int_1^5 3x\cos x\,dx$

$\lim_{n\to\infty}\sum_{n=1}^{n} 3\left(1+\frac{4}{n}k\right)\cos\left(1+\frac{4}{n}k\right)\frac{4}{n}$

D) $\int_0^4 3x\cos(1+x)\,dx = \lim_{n\to\infty} 3\left(\frac{4}{n}k\right)\cos\left(1+\frac{4}{n}k\right)\frac{4}{n}$

2. Use the given areas to evaluate the following integrals.



a) $\int_a^b f(x)\,dx = 0.8$

b) $\int_b^c f(x)\,dx = -2.6$

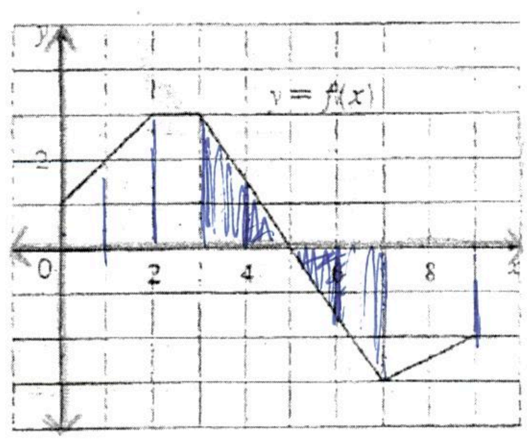
c) $\int_a^c f(x)\,dx = -1.8$

$0.8 - 2.6$

d) $\int_a^d f(x)\,dx = -0.3$

$0.8 - 2.6 + 1.5$

3. f(x) is given to the right. Use area to evaluate the following integrals.



a) $\int_0^2 f(x)\,dx = 4$

b) $\int_0^5 f(x)\,dx = 10$

$7 + 3$

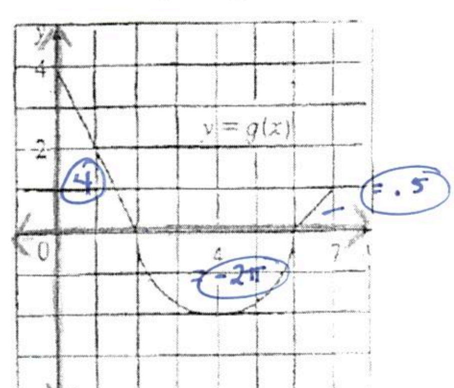
c) $\int_5^7 f(x)\,dx = -3$

d) $\int_0^9 f(x)\,dx = 2$

$= \int_0^3 f(x)\,dx + \int_7^9 f(x)\,dx$

$= 7 - 5$

4. The graph of g(x) consists of two straight lines and a semicircle. Use area to evaluate the following integrals.



a) $\int_0^2 g(x)\,dx = 4$

b) $\int_2^6 g(x)\,dx = -2\pi$

c) $\int_0^7 g(x)\,dx$

$\frac{9}{2} - 2\pi$

d) $\int_7^0 g(x)\,dx$

$2\pi - \frac{9}{2}$