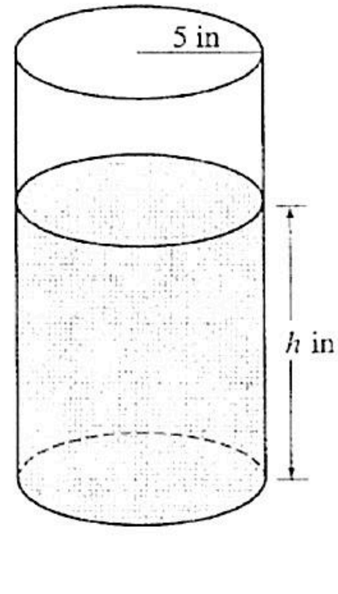
AP Differential Free Response Packet name *Key*

1) A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure. Let h be the depth of the coffee in the pot, measured in inches, where h is a function of time t, measured in seconds. The volume V of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)



$\frac{dV}{dt} = -5\pi\sqrt{h}$

$\frac{dr}{dt} = 0$

h is changing, r is constant

(a) Show that $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$.

$$\frac{dV}{dt} = \pi r^2 \cdot \frac{dh}{dt} + h \cdot 2\pi r \frac{dr}{dt}$$

$$-5\pi\sqrt{h} = \pi(25)\frac{dh}{dt} + \cancel{h \cdot 2\pi \cdot 5} \cdot 0$$

$$\frac{dh}{dt} = \frac{-5\pi\sqrt{h}}{25\pi} = -\frac{\sqrt{h}}{5}$$

(b) Given that $h = 17$ at time $t = 0$, solve the differential equation $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$ for h as a function of t.

$$\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{5}\, dt$$

$$\int h^{-1/2} dh = -\frac{1}{5}t + c$$

$$2\sqrt{h} = -\frac{1}{5}t + c$$

$$2\sqrt{17} = -\frac{1}{5}(0) + c$$

$$c = 2\sqrt{17}$$

$$2\sqrt{h} = -\frac{1}{5}t + 2\sqrt{17}$$

$$\sqrt{h} = -\frac{1}{10}t + \sqrt{17}$$

$$\boxed{h = \left(-\frac{1}{10}t + \sqrt{17}\right)^2}$$

(c) At what time t is the coffeepot empty?

The coffee pot is empty when $h = 0$

$$0 = \left(-\frac{1}{10}t + \sqrt{17}\right)^2$$

$$-\frac{1}{10}t + \sqrt{17} = 0$$

$$\frac{1}{10}t = \sqrt{17}$$

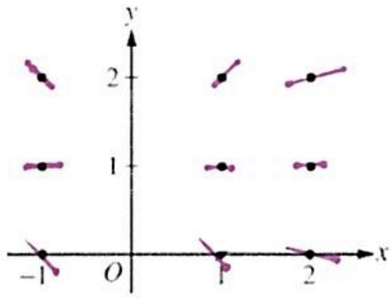
$$\boxed{t = 10\sqrt{17} \text{ seconds}}$$

2008 AP® CALCULUS AB FREE-RESPONSE QUESTIONS

5. Consider the differential equation $\frac{dy}{dx} = \frac{y-1}{x^2}$, where $x \neq 0$.

(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.

(Note: Use the axes provided in the exam booklet.)



(x,y)	$\frac{dy}{dx}$
$(1,0)$	$\frac{-1}{1} = -1$
$(2,0)$	$\frac{-1}{4}$
$(1,2)$	$\frac{1}{1}$
$(2,2)$	$\frac{1}{4}$

(b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(2) = 0$.

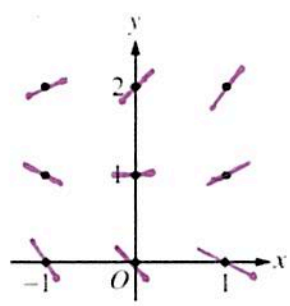
(c) For the particular solution $y = f(x)$ described in part (b), find $\lim_{x\to\infty} f(x)$.

2007 AP® CALCULUS AB FREE-RESPONSE QUESTIONS (Form B)

5. Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}x + y - 1$.

(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.

(Note: Use the axes provided in the exam booklet.)



(x,y)	$\frac{dy}{dx}$
$(0,0)$	-1
$(1,0)$	$\frac{1}{2} + 0 - 1 = -1/2$
$(-1,0)$	$-\frac{1}{2} + 0 - 1 = -3/2$
$(0,1)$	$0 + 1 - 1 = 0$
$(1,1)$	$\frac{1}{2} + 1 - 1 = 1/2$
$(-1,1)$	$-\frac{1}{2} + 1 - 1 = -1/2$
$(1,2)$	$\frac{1}{2} + 2 - 1 = 3/2$

(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Describe the region in the xy-plane in which all solution curves to the differential equation are concave up.

(c) Let $y = f(x)$ be a particular solution to the differential equation with the initial condition $f(0) = 1$. Does f have a relative minimum, a relative maximum, or neither at $x = 0$? Justify your answer.

5. At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that W will satisfy the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010.

(a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

(b) Find $\frac{d^2W}{dt^2}$ in terms of W. Use $\frac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \frac{1}{4}$.

(c) Find the particular solution $W = W(t)$ to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$.

2007 #5 Form B

a) see paper

b) $\frac{d^2y}{dx^2} = \frac{1}{2} + \frac{dy}{dx} = \frac{1}{2} + \frac{1}{2}x + y - 1 = \boxed{\frac{1}{2}x + y - \frac{1}{2}}$

$\frac{1}{2}x + y - \frac{1}{2} > 0$ when y is concave up.

$y > -\frac{1}{2}x + \frac{1}{2}$ $\therefore$ all points above the line $y = -\frac{1}{2}x + \frac{1}{2}$ will be concave up.

c) $\left.\frac{dy}{dx}\right|_{(0,1)} = \frac{1}{2}(0) + 1 - 1 =$

Since $\left.\frac{dy}{dx}\right|_{(0,1)} = 0$, there is a critical value at $x=0$

$\left.\frac{d^2y}{dx^2}\right|_{(0,1)} = \frac{1}{2}(0) + 1 - \frac{1}{2} = \frac{1}{2}$ Since $\frac{d^2y}{dx^2} > 0$, the function is concave up at $(0,1)$

$\therefore$ (0,1) is a relative minimum for $y = f(x)$.

2008 #5

a) see paper

b) $\int \frac{dy}{y-1} = \int \frac{dx}{x^2}$ (2,0)

$\ln|y-1| = -\frac{1}{x} + C$

$\ln|-1| = -\frac{1}{2} + C$

$C = \frac{1}{2}$

$\ln|y-1| = -\frac{1}{x} + \frac{1}{2}$

$e^{-\frac{1}{x}+\frac{1}{2}} = |y-1| \rightarrow y-1 = \pm e^{-\frac{1}{x}+\frac{1}{2}}$

$y = 1 \pm e^{\frac{-1}{x}+\frac{1}{2}}$

$y = -e^{-\frac{1}{x}+\frac{1}{2}} + 1$

c) $\lim_{x\to\infty} \left(1 - e^{-\frac{1}{x}+\frac{1}{2}}\right) =$

$\lim_{x\to\infty} \left(\frac{-1}{x}\right) = 0 \quad \therefore \lim_{x\to\infty} \left(1 - e^{-\frac{1}{x}+\frac{1}{2}}\right) = 1 - e^{1/2}$

2011 #5

$\frac{dw}{dt} = \frac{1}{25}(w-300)$ for $0 \le t \le 20$

$W(0) = 1400$

$t=0 \rightarrow 2010$

a) $\left.\frac{dw}{dt}\right|_{(0,1400)} = \frac{1}{25}(1400-300) = 44$

$y - 1400 = 44x$

$y(\frac{1}{4}) = 11 + 1400 = 1411$

$W(\frac{1}{4}) \approx 1411$ tons of solid waste

b) $\frac{d^2w}{dt^2} = \frac{1}{25}\left(\frac{dw}{dt}\right) = \frac{1}{25}\left(\frac{1}{25}(w-300)\right) = \frac{w-300}{625}$

$\left.\frac{d^2w}{dt^2}\right|_{(0,1400)} = \frac{1400-300}{625} > 0$ on $0 \le t \le \frac{1}{4}$

$\therefore$ $w(t)$ is concave up on $0 \le t \le \frac{1}{4}$ making part a an underestimate

c) $\int \frac{dw}{w-300} = \int \frac{1}{25}\, dt$

$\ln|w-300| = \frac{1}{25}t + C$

$\ln|1100| = \frac{1}{25}(0) + C$

$C = \ln|1100|$

$\ln|w-300| = \frac{1}{25}t + \ln|1100|$

$w = 1100e^{\frac{1}{25}t} + 300$