Notes: Antiderivatives and Indefinite Integration

An antiderivative is a solution to a differential equation. It "undoes" the derivative.
To find an antiderivative, or indefinite integral, the function must be continuous.

$$\frac{dy}{dx} = f(x) \quad \Rightarrow \quad dy = f(x)dx \quad \Rightarrow \quad y = \int f(x)dx + C$$

* The C is called the constant of integration. It is required on all indefinite integrals because the derivative of a constant is always 0. We call the solution with C the "general solution".

example: $f(x) = x^2$ $\quad g(x) = x^2 + 2$ $\quad h(x) = x^2 + \pi$

The derivative of each of these functions is $f'(x) = 2x$. If we are working backwards, there is no way to tell if the original function had a constant without knowing a point on the original function.

function	**derivative**
x^n	nx^{n-1}
sin x	cos x
cos x	-sin x
tan x	$\sec^2 x$
csc x	-csc x cot x
sec x	sec x tan x
cot x	$-\csc^2 x$
e^x	e^x
ln x	$\frac{1}{x}$
a^x	$a^x \ln a$

function	**integral**
x^n	$\frac{x^{n+1}}{n+1} + c$
cos x	sin x+C
sin x	-cosx+C
$\sec^2 x$	tan x+C
csc x cot x	-csc x+C
sec x tan x	sec x+C
$\csc^2 x$	-cot x + C
e^x	e^x + C
$\frac{1}{x}$	ln |x|+C
a^x	$\frac{a^x}{\ln a}$ +C

Power Rule for Integrals Examples: $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

1a. $\int 3x\,dx = \frac{3}{2}x^2 + c$

d. $\int 1\,dx =$ x

b. $\int \frac{1}{x^3}dx = \int x^{-3}dx = \frac{x^{-2}}{-2} + c = -\frac{1}{2x^2} + c$

e. $\int (3x^4 + 5x^3 - 7x^2)\,dx =$

$\frac{3}{5}x^5 + \frac{5}{4}x^4 - \frac{7}{3}x^3 + c$

c. $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}x^{\frac{3}{2}} + c$

2a. $\int \frac{x+1}{\sqrt{x}}dx =$

$\int \left(\frac{x}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{1}{2}}} \right) dx = \int (x^{\frac{1}{2}} + x^{-\frac{1}{2}})dx = \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c$

c. $\int (1 + \tan^2 x)\,dx = \int \sec^2 x\,dx = \tan x + c$

b. $\int \frac{\sin x}{\cos^2 x}dx = \int \left(\frac{\sin x}{\cos x} \bullet \frac{1}{\cos x} \right) dx = \int \tan x \sec x\,dx = \sec x + c$

3. Find f(x) using the given information:

$f''(x) = x^2,\ f'(0) = 6,\ f(0) = 3$

$f'(x) = \int f''(x)dx = \int x^2 dx = \frac{x^3}{3} + c$

$6 = \frac{0^3}{3} + c$, so c = 6

$f'(x) = \frac{x^3}{3} + 6;\ \int f'(x)dx = f(x),\ so\ \int (\frac{x^3}{3} + 6)dx = \frac{x^4}{12} + 6x + c$

$3 = \frac{0^4}{12} + 6(0) + c,\ so\ c = 3$

$f(x) = \frac{x^4}{12} + 6x + 3$

4. A ball is thrown upward from an initial height of 80 feet with initial velocity 64 ft/sec. Acceleration due to gravity is -32 ft/sec^2.

a. Find position s as a function of t.

$a(t) = -32dt$

$v(t) = \int a(t)dt = \int -32dt = -32t + C$

$v(0) = 64, so\ \ 64 = -32(0) + C$

$c = 64$

$v(t) = -32t + 64$

$s(t) = \int v(t)dt = \int (-32t + 64)dt = -16t^2 + 64t + C$

$s(0) = 80,\ \ so\ \ \ 80 = -16(0)^2 + 64(0) + C$

$C = 80$

$s(t) = -16t^2 + 64t + 80$

b. When does the ball hit the ground?

$Since\ the\ ball\ will\ hit\ the\ ground\ when\ s(t) = 0,$

$0 = -16t^2 + 64t + 80$

$0 = -16(t-5)(t+1)$

$t = 5\sec onds$

c. What was the maximum height of the ball?

The ball will reach its max height when v(t) = 0

$v(t) = -32t + 64$

$0 = -32(t-2)$

The ball will reach its max height when t = 2.

$s(2) = -16(2)^2 + 64(2) + 80$

$s(2) = 144\ \ \ ft.$