NATIONAL MATH + SCIENCE INITIATIVE

Key

Fundamental Theorem of Calculus

You will recall that $\int_a^b f'(x)\,dx = f(b) - f(a)$.

This formula can be rewritten to show that $f(b) = f(a) + \int_a^b f'(x)\,dx$. This means that a final value is equal to the initial value + net change.

Accumulation

One of the key ideas in mathematics is that $rate \times time = amount$. If a printer that is rated at 8 pages per minute prints for 3 minutes, it can print $8\,pages/\min \times 3\min = 24\,pages$. Notice how the units work out correctly. While this example has a constant rate, it is possible to have a rate that varies according to some function. To accumulate a rate of change over time, we use a definite integral: $Amount = \int_{begining\ time}^{ending\ time} Rate\ dt$.

Over the past decade the AP Exam has had many problems involving the concept of adding a quantity at one rate while subtracting a quantity using another rate. Usually, there is some initial amount to consider. This gives the following conceptual formula

$Current\ Amount = Initial\ Amount + \int_{time1}^{time2} additionrate\ dt - \int_{time1}^{time2} subtractionrate\ dt$. Also, you may be asked to find when the amount reaches a maximal or minimal amount. This usually involves the Fundamental Theorem of Calculus.

As with other types of problems, accumulation problems can be presented in graphical, numerical, or analytical formats.

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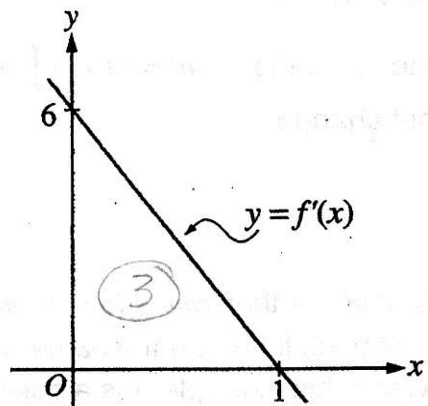
Multiple Choice Examples

1. 2003 AB22 - No Calculator Allowed



$5 + \int_0^1 f'(x)\,dx$

$= 5 + 3$

22. The graph of f', the derivative of f, is the line shown in the figure above. If $f(0) = 5$, then $f(1) =$

(A) 0 (B) 3 (C) 6 (D) 8 (E) 11

$350 + \int_0^5 (-110e^{-.4t}\,dt)$

2. 2003 AB84 - Calculator Active

84. A pizza, heated to a temperature of 350 degrees Fahrenheit (°F), is taken out of an oven and placed in a 75°F room at time $t = 0$ minutes. The temperature of the pizza is changing at a rate of $-110e^{-0.4t}$ degrees Fahrenheit per minute. To the nearest degree, what is the temperature of the pizza at time $t = 5$ minutes?

(A) 112°F (B) 119°F (C) 147°F (D) 238°F (E) 335°F

3. 2003 AB91 - Calculator Active

91. A particle moves along the x-axis so that at any time $t > 0$, its acceleration is given by $a(t) = \ln\left(1 + 2^t\right)$. If the velocity of the particle is 2 at time $t = 1$, then the velocity of the particle at time $t = 2$ is

(A) 0.462 (B) 1.609 (C) 2.555 (D) 2.886 (E) 3.346

$2 + \int_1^2 \ln(1+2^t)\,dt$

4. 2003 BC80 - Calculator Active

80. Insects destroyed a crop at the rate of $\dfrac{100e^{-0.1t}}{2 - e^{-3t}}$ tons per day, where time t is measured in days. To the nearest ton, how many tons did the insects destroy during the time interval $7 \le t \le 14$?

(A) 125 (B) 100 (C) 88 (D) 50 (E) 12

$\int_7^{14} \dfrac{100e^{-.1t}}{2 - e^{-3t}}\,dt$

5. 2003 BC87 - Calculator Active

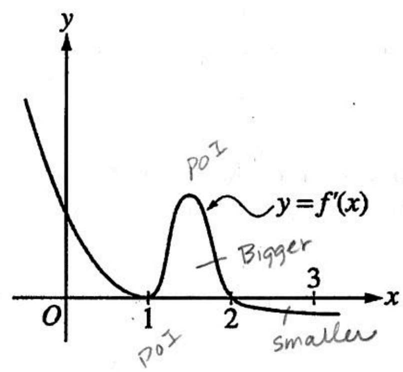
87. A particle moves along the x-axis so that at any time $t \geq 0$, its velocity is given by $v(t) = \cos(2 - t^2)$. The position of the particle is 3 at time $t = 0$. What is the position of the particle when its velocity is first equal to 0 ?

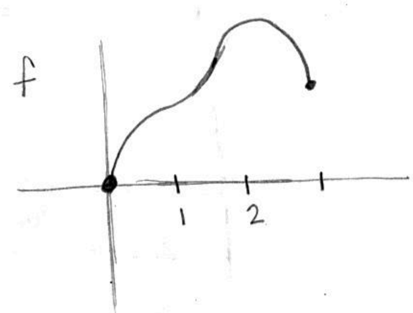
(A) 0.411 (B) 1.310 (C) 2.816 (D) 3.091 (E) 3.411

$v(t) = 0$
when $t = .655$

$3 + \int_0^{.655} v(t)\,dt$

6. 2003 BC90 - Calculator Active





90. The graph of f', the derivative of the function f, is shown above. If $f(0) = 0$, which of the following must be true?

✗I. $f(0) > f(1)$
✓II. $f(2) > f(1)$
✗III. $f(1) > f(3)$

$f(0) = 0$
$f(1) > f(0)$
$f(2) > f(1)$
$f(3) < f(2)$

(A) I only
(B) II only
(C) III only
(D) I and II only
(E) II and III only

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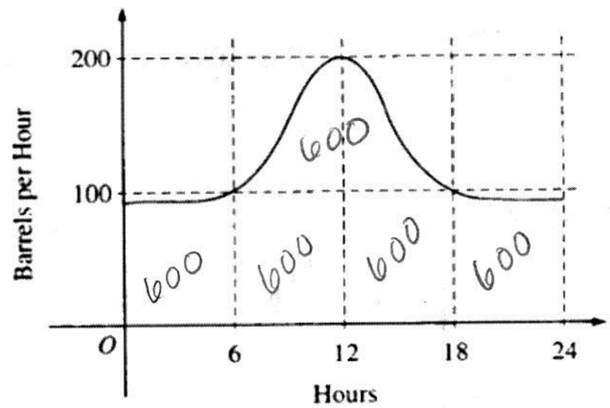
7. 1998 AB9 BC9 - No Calculator Allowed



9. The flow of oil, in barrels per hour, through a pipeline on July 9 is given by the graph shown above. Of the following, which best approximates the total number of barrels of oil that passed through the pipeline that day?

(A) 500 (B) 600 (C) 2,400 (D) 3,000 (E) 4,800

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Free Response

1. 2005 AB2 Calculator Allowed

The tide removes sand from Sandy Beach at a rate modeled by the function *R*, given by

$$R(t) = 2 + 5\sin\left(\frac{4\pi t}{25}\right)$$

A pumping station adds sand to the beach at a rate modeled by the function *S*, given by

$$S(t) = \frac{15t}{1+3t}$$

Both *R(t)* and *S(t)* have units of cubic yards per hour and *t* is measured in hours for $0 \le t \le 6$. At time *t* = *0*, the beach contains 2500 cubic yards of sand.

(a) How much sand will the tide remove from the beach during this 6-hour period? Indicate units of measure.

$\int_0^6 R(t)\,dt = 31.8159$ cubic yards of sand

(b) Write an expression for *Y(t)*, the total number of cubic yards of sand on the beach at time *t*.

$Y(t) = 2500 + \int_0^t [S(t) - R(t)]\,dt$

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(c) Find the rate at which the total amount of sand on the beach is changing at time $t = 4$.

$y'(4) = S(4) - R(4)$

$= -1.908 \ yd^3/hr$

(d) For $0 \le t \le 6$, at what time t is the amount of sand on the beach a minimum? What is the minimum value? Justify your answers.

t	$Y(t)$
0	2500
5.117	2492.369
6	2493.276

CV when $y'(t)=0$

@ $t = 5.117$

The amount of sand is at a minimum when $t = 5.117$ hrs.

The minimum is 2492.369 yds^3

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★ No

2. 2000 AB4 – Calculator Allowed

Water is pumped into an underground tank at a constant rate of 8 gallons per minute. Water leaks out of the tank at the rate of $\sqrt{t+1}$ gallons per minute, for $0 \le t \le 120$ minutes. At time *t = 0*, the tank contains 30 gallons of water.

(a) How many gallons of water leak out of the tank from time *t = 0* to *t = 3* minutes?

$$\int_0^3 \sqrt{t+1}\, dt \qquad u = t+1 \quad du = dt$$

$$\int_1^2 u^{1/2}\, du = \frac{2}{3} u^{3/2} \Big|_1^4 = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}$$ gallons of water leaked in the 1st 3 minutes.

(b) How many gallons of water are in the tank at time *t = 3* minutes?

$$30 - \int_0^{30} \sqrt{t+3}\, dt + 8 \cdot 3 = 54 - \frac{14}{3} = \frac{148}{3}$$ gallons

(c) Write an expression for *A(t)*, the total number of gallons of water in the tank at time *t*.

$$A(t) = 30 + \int_0^t 8\,dt - \int_0^t \sqrt{x+1}\,dx$$

or

$$A(t) = 30 + 8t - \frac{2}{3}(t+1)^{3/2} + \frac{2}{3}$$

(d) At what time *t*, for $0 \le t \le 120$, is the amount of water in the tank a maximum? Justify your answer.

t	$A(t)$
0	$30\ 2/3$
63	$193\ 1/3$
120	$103.\ 1/3$

$A'(t) = 0$ when $t = 63$

$A' = 8 - \sqrt{t+1}$

$8 = \sqrt{t+1}$

$64 - 1 = t$

$\therefore$ The max amount of water in the tank occurs @ $t = 63$ minutes.

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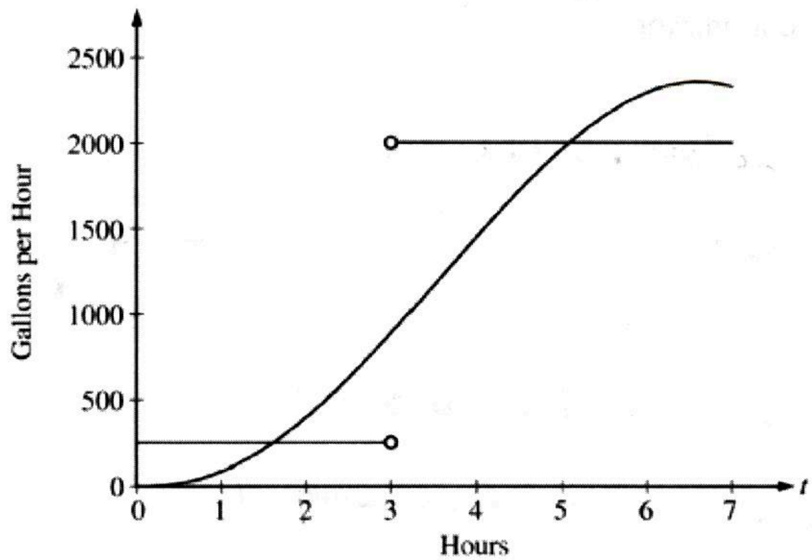
3. 2007 AB2 – Calculator allowed



The amount of water in a storage tank, in gallons, is modeled by a continuous function on the time interval $0 \le t \le 7$, where t is measured in hours. In this model, rates are given as follows:

(i) The rate at which water enters the tank is $f(t) = 100t^2 \sin\left(\sqrt{t}\right)$ gallons per hour for $0 \le t \le 7$.

(ii) The rate at which water leaves the tank is

$$g(x) = \begin{cases} 250 & for\, 0 \le t < 3 \\ 2000 & for\, 3 < t \le 7 \end{cases}$$ gallons per hour.

The graphs of *f* and *g*, which intersect at *t* = *1.617* and *t* = *5.076* are shown in the figure above. At time *t* = *0*, the amount of water in the tank is 5000 gallons.

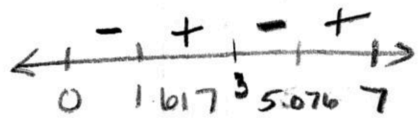
(a) How many gallons of water enter the tank during the time interval $0 \le t \le 7$? Round your answer to the nearest gallon.

$\int_0^7 f(t)\,dt \approx 8264$ gallons

(b) For $0 \le t \le 7$, find the time intervals during which the amount of water in the tank is decreasing. Give a reason for your answer.

Rate of amount of water in the tank

$= f(t) - g(t)$



$f(t) = g(t)$ when $t = 1.617$ & 5.076

The amount of water is decreasing when

$f(t) - g(t) < 0$ $\therefore$ $[0, 1.617) \cup (3, 5.076)$

(c) For $0 \le t \le 7$, at what time t is the amount of water in the tank greatest? To the nearest gallon, compute the amount of water at this time. Justify your answer.

t	$A(t)$
0	5000
1.617	Rel min X
3	Rel max $= 5000 + \int_0^3 f(t)\,dt - 3(250) = 5{,}126.591$
5.076	Rel min X
7	$A(3) + \int_3^7 f(t)\,dt - 2000(4) = 4513.807$

$\therefore$ The amount of water in the tank is greatest @ $t = 3$ hours when there are ≈ 5127 gallons of water in the tank.