Notes on FTC Part 2 and AROC vs Average Value

We know from part 1 of the fundamental theorem that:

$$\int_a^b f'(x)dx = f(b) - f(a)$$

Part 2 states that

$$\boxed{\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(t)dt\right] = f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)}$$

Let's Prove it:

$$F(x) = \int_{\sin x}^{\ln x} (t^2)dt$$ Find $F'(x)$

First, evaluate the integral:

$$F = \frac{t^3}{3}\Big|_{\sin x}^{\ln x} = \frac{(\ln x)^3}{3} - \frac{(\sin x)^3}{3}$$

Now derive.

$$F' = \frac{3(\ln x)^2}{3}\cdot\frac{1}{x} - \frac{3(\sin x)^2}{3}\cdot\cos x = (\ln x)^2\cdot\frac{1}{x} - \sin^2 x\cdot\cos x$$

It would have been much faster to find $F'(x)$ using the 2nd fundamental theorem instead of doing all that work.

Just plug in the limits, and multiply by the derivative of what you plug in. Subtract top – bottom. If either of the limits of integration is a constant, don't forget that the derivative of a constant is 0.

Examples

$$\frac{d}{dx}\int_x^{\pi} \sin t\,dt =$$

$$\sin\pi\cdot 0 - \sin x\cdot 1$$

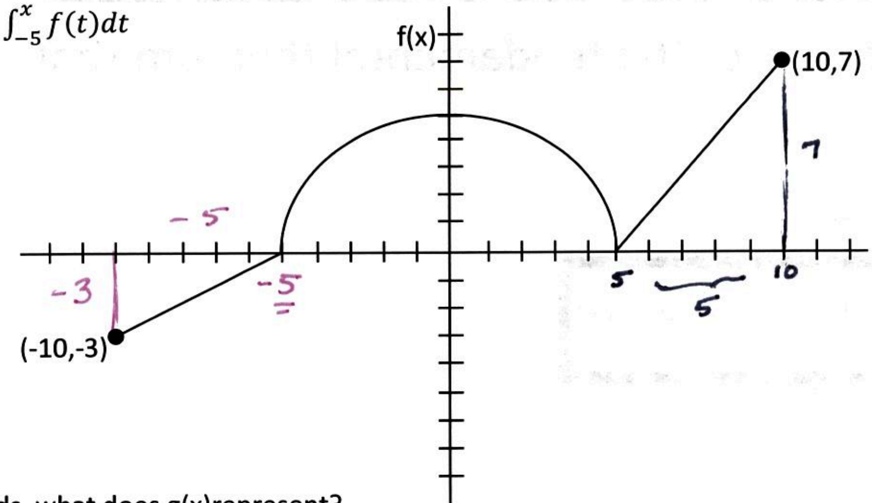
$$= -\sin x$$

$$\frac{d}{dx}\int_1^{x^2} \sqrt{t^3 - t}\,dt =$$

$$\sqrt{(x^2)^3 - x^2}\cdot 2x - \sqrt{1-1}\cdot 0$$

$$= 2x\sqrt{x^6 - x^2}$$

Given that the following function, f(x), is comprised of two line segments and a semicircle, answer the following questions. $g(x) = \int_{-5}^{x} f(t)dt$



1. In your own words, what does g(x)represent?

The signed area between f(x) + the x-axis from t = -5 to t = x

2 What is g'(x)? $f(x)$ 2nd FTC $g' = f(x)\cdot 1 - f(-5)\cdot 0 = f(x)$

3. In your own words, what does g''(x)represent?

f'(x), or the slope of f(x)

4. Find g(-5) = 0 $\int_{-5}^{-5} f(t)dt$

5. Find g(5)= $\frac{25\pi}{2}$ $\int_{-5}^{5} f(t)dt$

6. Find g(-10) = 7.5 $\int_{-5}^{-10} f(t)dt$

7. Find g(10) = $\frac{25\pi + 35}{2}$ $\int_{-5}^{10} f(t)dt$

8. Find g'(-10) = -3 $g'(-10) = f(-10)$

9. Find g'(0) = 5 $g'(0) = f(0)$

10. Find g''(0) = 0 $g''(0) = f'(0)$

11. Find g''(6) = $7/5$ $g''(6) = f'(6)$

12. Find g''(-5) = DNE (Sharp turn) $g''(-5) = f'(-5)$

13. On what intervals would g(x) be decreasing? Justify your conclusions.

g is increasing when $g' = f \geq 0$; ∴ g is increasing on [-5, 10]

14. Find the x-values of any relative extrema for g(x). Justify your conclusions.

g has a relative max when $g' = f$ changes from + to −, ∴ no rel max

g has a relative min when $g' = f$ changes

15. On what intervals is g(x) concave up? Justify your conclusions.

g is concave up when $g' = f$ is increasing. ∴ g is concave up on $(-10,-3) \cup (-3,0) \cup (5,10)$

16. Find the x-values of any points of inflection for g(x). Justify your conclusions.

g has PoI when $g' = f$ has a relative extrema ∴ @ x = 0, 5

17. Find the range of g(x). Justify your conclusions. $\left[0, \frac{25\pi + 35}{2}\right]$

↓
Absolute
Min to
Absolute Max

x	g(x)
-10	7.5
-5	0
5	$25\pi/2$
10	$\frac{25\pi+35}{2}$

What is the difference between AROC and Average Value?

The average value of a function is $\frac{1}{b-a}\int_a^b f(x)dx$

AROC is the average rate of change of a function and is $\frac{f(b)-f(a)}{b-a}$.

You need to be very careful that you use these formulas correctly, especially when working velocity.

$\frac{1}{b-a}\int_a^b v(t)dt$ is the average velocity of a function, and using FTC and the fact that $v(t)=x'(t)$

$$\frac{1}{b-a}\int_a^b v(t)dt=\frac{1}{b-a}\int_a^b x'(t)dt=\frac{x(b)-x(a)}{b-a}$$

Example: The velocity of a particle is given by $v(t)=t^2-3t$.

Find the average velocity of the particle over the time interval [1,4].

$$\frac{1}{4-1}\int_1^4 v(t)dt = \frac{\left.\frac{t^3}{3}-\frac{3t^2}{2}\right|_1^4}{3} = \frac{\left(\frac{64}{3}-\frac{48}{2}\right)-\left(\frac{1}{3}-\frac{3}{2}\right)}{3} = \frac{\frac{42}{2}-\frac{45}{2}}{3}$$

$$= -\frac{3}{2}\cdot\frac{1}{3} = -\frac{1}{2}$$

Find the average acceleration of the particle over the time interval [1,4].

OPTION 1

$a(t)=v'(t)=2t-3$

$$\frac{1}{3}\int_1^4 (2t-3)dt = \frac{(8-3)-(2-3)}{3}$$

$$= 2$$

OPTION 2

$$\frac{v(4)-v(1)}{4-1} = \frac{(16-12)-(1-3)}{3}$$

$$= \frac{4+2}{3}$$

$$= 2$$