Integration Using U-substitution (undoing the chain rule)

Chain Rule Review:

$f(x) = (3x^2+4)^3$

$f'(x) = 3(3x^2+4)^2 \cdot 6x$

$= 18x(3x^2+4)^2$

$\therefore \int 18x(3x^2+4)^2\,dx = (3x^2+4)^3 + C$

S: $(\quad)^3$ ⋮ $3(\quad)^2$

C: $(3x^2+4)$ ⋮ $6x$

→ In order to "undo" the chain rule, we first need to identify the "chocolate". Let u = chocolate. – This is the most important step. u is usually the expression "inside" another function.

$u = 3x^2+4$

→ Now take the derivative of u with respect to x.

$\frac{du}{dx} = 6x$

→ Now solve for dx.

$dx = \frac{du}{6x}$

→ Substitute.

$\int \cancel{18x}^{\,3} (u)^2 \cdot \frac{du}{\cancel{6x}}$

→ Simplify & Integrate

$\int 3u^2\,du = u^3 + C$

→ Plug u back in & you are done!

$\int 18x(3x^2+4)^2\,dx = (3x^2+4)^3 + C$

Example: $\int x\sqrt{3x^2+5}\,dx$

$u = 3x^2+5$

$\frac{du}{dx} = 6x$

$dx = \frac{du}{6x}$

$$= \int \cancel{x} \cdot u^{1/2} \cdot \frac{du}{6\cancel{x}}$$

* If the x's don't cancel, you did something wrong.

$$= \frac{1}{6}\int u^{1/2}\,du$$

$$= \frac{1}{6} \cdot \frac{2}{3} u^{3/2} + C$$

$$= \frac{1}{9}(3x^2+5)^{3/2} + C$$

Example: $\int \frac{x^2}{(x^3-4)^4}\,dx$

$u = x^3-4$

$\frac{du}{dx} = 3x^2$

$dx = \frac{du}{3x^2}$

$$= \int \frac{1\cancel{x^2}}{u^4} \cdot \frac{du}{3\cancel{x^2}}$$

$$= \frac{1}{3}\int u^{-4}\,du$$

$$= \frac{1}{3} \cdot \frac{u^{-3}}{-3} + C$$

$$= -\frac{1}{9}(x^3-4)^{-3} + C$$

or

$$\frac{-1}{9(x^3-4)^3} + C$$

$$\int \sin(5x)\,dx$$

$u = 5x$

$\frac{du}{dx} = 5$

$dx = \frac{du}{5}$

$= \int \sin u \cdot \frac{du}{5}$

$= \frac{1}{5} \int \sin u \, du$

$= -\frac{1}{5} \cos u + C$

$= -\frac{1}{5} \cos(5x) + C$

$$\int 3e^{2x}\,dx$$

$u = 2x$

$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

$= 3 \int e^u \cdot \frac{du}{2}$

$= \frac{3}{2} \int e^u \, du$

$= \frac{3}{2} e^u + C$

$= \frac{3}{2} e^{2x} + C$

Example: $\int x \cdot 2^{3x^2} dx$

$u = 3x^2$

$\frac{du}{dx} = 6x$

$dx = \frac{du}{6x}$

$$= \int \cancel{x} \cdot 2^u \cdot \frac{du}{6\cancel{x}}$$

$$= \frac{1}{6} \int 2^u \, du$$

$$= \frac{1}{6} \cdot \frac{2^u}{\ln 2} + C$$

$$= \frac{2^{3x^2}}{6\ln 2} + C$$

Example: $\int_{0=x}^{2=x} x(x^2+4)^4 \, dx$

$u = x^2+4$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$$= \int_4^8 \cancel{x}(u)^4 \cdot \frac{du}{2\cancel{x}}$$

$$= \frac{1}{2} \int_4^8 u^4 \, du$$

$$= \frac{1}{2} \cdot \frac{u^5}{5} \Big|_4^8$$

$$= \frac{8^5}{10} - \frac{4^5}{10}$$

$$= \frac{4^5 \cdot 2^5 - 4^5}{10}$$

$$= \frac{4^5(2^5-1)}{10}$$

$$= \frac{4^5 \cdot 31}{10}$$

* now that we switched to u, our limits of integration must be u!

if $x=2$, $u=8$

if $x=0$, $u=4$

Example: $\int_0^{\pi} \cos\left(\frac{x}{2}\right) dx$

$u = \frac{x}{2}$

$\frac{du}{dx} = \frac{1}{2}$

$dx = 2du$

when $x = 0$, $u = 0$

when $x = \pi$, $u = \frac{\pi}{2}$

$$= \int_0^{\frac{\pi}{2}} \cos u \cdot 2du$$

$$= 2\int_0^{\frac{\pi}{2}} \cos u \, du$$

$$= 2 \sin u \Big|_0^{\frac{\pi}{2}}$$

$$= 2 \sin \frac{\pi}{2} - 2 \sin 0$$

$$= 2(1) - 2(0)$$

$$= 2$$

Example: $\int_1^{-1} \frac{2x^3 - 3x^2}{(x^4 - 2x^3)^2} dx$

$u = x^4 - 2x^3$

$\frac{du}{dx} = 4x^3 - 6x^2$

$dx = \frac{du}{2(2x^3 - 3x^2)}$

when $x = 1$, $u = -1$

when $x = -1$, $u = -4 + 6$
$= 2$

$$= \int_{-1}^{2} \frac{\cancel{2x^3 - 3x^2}}{u^2} \cdot \frac{du}{2\cancel{(2x^3 - 3x^2)}}$$

$$= \frac{1}{2} \int_{-1}^{2} u^{-2} du$$

$$= \frac{1}{2} \cdot \frac{u^{-1}}{-1} \Big|_{-1}^{2}$$

$$= -\frac{1}{2} \cdot \frac{1}{2} - \left(\frac{1}{2} \cdot \frac{-1}{-1}\right) = \frac{1}{4}$$